Introduction to Lie Algebras and Their Representations
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Unofficial lecture notes - University of Cambridge
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Contents

Reading to complement course material

Jacobson, N. (1979), *Lie algebras*, Dover Publications.

Kac, V. (1994), *Infinite dimensional Lie algebras*, 3 edn, Cambridge University Press.

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1 Introduction

1.1 Motivation

Definition 1.1: A *linear algebraic group* is a subgroup of the general linear group GL_n of $n \times n$ matrices where the matrix coefficients fulfill certain polynomial equations.

Example 1.2: The *upper triangle matrices*

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Other examples are

$$\begin{aligned} SL_n &= \{A \in \mathrm{Mat}_n \mid \det A = 1\}, \\ SO_n &= \{A \in SL_n \mid AA^T = I\}, \\ O_n &= \{A \in GL_n \mid AA^T = I\}, \\ SP_{2n} &= \{A \in GL_n \mid M^T A^T M A = I\}, \quad M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \end{aligned}$$

Remark 1.3: There is an intrinsic characterization of linear algebraic groups as *affine algebraic groups*, i.e. groups which are affine algebraic varieties and where multiplication and inverse are morphisms of algebraic varieties.

Consider $G = SL_n$. If

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \text{higher order terms} \in SL_2, \ |\varepsilon| \ll 1,$$

then

$$\begin{aligned} 1 = \det g &= \det\left(\begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1+\varepsilon d \end{pmatrix} + \text{higher order terms} \right) \\ &= (1+\varepsilon a)(1+\varepsilon d) - \varepsilon^2 cb + \text{higher order terms} \\ &= 1 + \varepsilon(a+d) + \text{higher order terms}. \end{aligned}$$

Thus, $\det g = 1$ if and only if $a + d = 0$.

We can make this notion of vanishing higher order terms more precise by defining the *dual numbers* as

$$E = \mathbb{C}[\varepsilon]/\varepsilon^2 = \{a + b\varepsilon \mid a, b \in \mathbb{C}\}.$$

This omits the structure of a ring. Consider $G(E) = \{A \in \mathrm{Mat}_n(E) \mid A \in G \subset GL_n\}$, the matrices over the dual numbers which satisfy the polynomial equations defining the linear algebraic group G. E.g.

$$SL_2(E) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in E: \ \alpha\delta - \beta\gamma = 1 \right\}.$$

By letting $\varepsilon \mapsto 0$ we obtain a map $E \to \mathbb{C}$ which extends to a map

$$\pi\colon G(E) \to G,\ A + B\varepsilon \mapsto A.$$

Definition 1.4: We define the *Lie algebra* of G, denoted $\mathrm{Lie}(G)$, as the preimage

$$\mathfrak{g} := \pi^{-1}(I) = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid I + \varepsilon X \in G(E)\}.$$

Example 1.5: For example $\mathfrak{sl}_2 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{C}) \mid a + d = 0\}$.

Remark 1.6: $I + X\varepsilon$ represents an 'infinitesimal change' at I in the direction X, i.e. the *germ* of a curve $\mathrm{Spec}[[\varepsilon]] \to G$.

Exercise 1.7: Show $G(E) = TG$, the *tangent bundle* to G, and $\mathfrak{g} = T_I G$, the *tangent space* to G at I.

Example 1.8:

(i) Let $G = GL_n = \left\{A \in \mathrm{Mat}_n \mid A^{-1} \text{ exists}\right\}$. Then

$$\begin{aligned} G(E) &= \left\{\tilde{A} \in \mathrm{Mat}_n(E) \mid \tilde{A}^{-1} \text{ exists}\right\} \\ &= \left\{A + B\varepsilon \mid A, B \in \mathrm{Mat}_n(\mathbb{C}),\ A^{-1} \text{ exists}\right\}, \end{aligned}$$

as $(A + B\varepsilon)(A^{-1} - A^{-1}BA^{-1}\varepsilon) = I$. So $\mathrm{Lie}(GL_n) = \mathrm{Mat}_n(\mathbb{C})$.

(ii) Let $G = SL_n(\mathbb{C})$. Then

$$\begin{aligned} \det(I + \varepsilon X) &= \det((\delta_{ij} + \varepsilon x_{ij})_{i,j}) \\ &= (1 + \varepsilon x_{11}) \cdot \ldots \cdot (1 + \varepsilon x_{nn}) \\ &= 1 + \varepsilon \mathrm{tr}(X). \end{aligned}$$

From this we conclude that

$$\mathfrak{sl}_n = \{X \in \mathrm{Mat}_n \mid \mathrm{tr}(X) = 0\}.$$

(iii) Let $G = O_n(\mathbb{C}) = \left\{A \mid AA^T = I\right\}$,

$$\begin{aligned} \Rightarrow \quad \mathfrak{g} &= \left\{X \in \mathrm{Mat}_n(\mathbb{C}) \mid (I + \varepsilon X)(I + \varepsilon X)^T = I\right\} \\ &= \left\{X \in \mathrm{Mat}_n(\mathbb{C}) \mid I + \varepsilon(X + X^T) = I\right\} \\ &= \left\{X \in \mathrm{Mat}_n(\mathbb{C}) \mid X + X^T = 0\right\}. \end{aligned}$$

Notice, that as $2 \neq 0$, we have $tr(X) = 0$, so this is also the Lie algebra of SO_n, denoted by $\mathfrak{so}_n$.

Remark 1.9: This leads us to the question, what structure we have in $\mathfrak{g}$ coming from G being a group? Note that in E we have $(I + A\varepsilon)(I + B\varepsilon) = I + (A + B)\varepsilon$, which has nothing to do with multiplication. Multiplication is a map $G \times G \to G$. Consider instead the map $G \times G \to G$ given by to commutator in a group, $(P, Q) \mapsto PQP^{-1}Q^{-1}$. If we look at this infinitesimally, we obtain a map $T_I G \times T_I G \to T_I G$, write $P = I + a\varepsilon$ and $Q = I + B\delta$, where $\varepsilon^2 = \delta^2 = 0$ but $\varepsilon\delta \neq 0$. Remember that $(I + a\varepsilon)^{-1} = I - A\varepsilon$. Then we have $PQP^{-1}Q^{-1} = I + (AB - BA)\varepsilon\delta$, which is the "shadow" of multiplication we will use. So for the Lie algebra of an algebraic groups, we define $[A, B] = AB - BA$, the *Lie bracket* of $\mathfrak{g}$.

Exercise 1.10: Show that:

(i) Show that $(PQP^{-1}Q^{-1})^{-1} = QPQ^{-1}P^{-1}$ implies $[A,B] = -[B,A]$, for all $A, B \in \mathfrak{g}$ (*skew symmetry*).

(ii) Multiplication in G is associative implies

$$0 = [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \ (\textit{Jacobi identity}).$$

Solution:

(i) Take P, Q as in 1.9, then as before

$$\begin{aligned}[B,A] \leftrightarrow QPQ^{-1}P^{-1} &= (PQP^{-1}Q^{-1})^{-1} = (I + (AB - BA)\varepsilon\delta)^{-1} \\ &= I + (-(AB-BA))\varepsilon\delta \leftrightarrow -[A,B],\end{aligned}$$

and so $[A,B] = -[B,A]$.
Remark: Since we already have $[A,B] = AB - BA$, we have the much easier and more obvious proof $[A,B] = AB - BA = -(BA - AB) = -[B,A]$.

(ii) By simple calculation, we obtain

$$\begin{aligned}[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] &= [XY - YX, Z] + [YZ - ZY, X] + [ZX - XZ, Y] \\ &= XYZ - YXZ - ZXY + ZYX + YZX - ZYX \\ &\quad - XYZ + XZY + ZXY - XZY - YZX + YXZ = 0.\end{aligned}$$

□

1.2 Definition of Lie algebras and basic properties

Definition 1.11: Let k be a field, $\operatorname{char} k \neq 2, 3$. A *Lie algebra* $\mathfrak{g}$ is a k-vector space equipped with a bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the so-called *Lie bracket*, such that

(i) $[X,Y] = -[Y,X]$, *skew symmetry*, and

(ii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$, the *Jacobi identity*.

A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Lie subalgebra* if $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$, i.e. for all $x, y \in \mathfrak{h}$, $[x,y] \in \mathfrak{h}$.

The previously defined Lie algebras of an algebraic groups satisfy these properties, as shown in 1.10.

Example 1.12: We have the following examples of Lie algebras:

(i) For any vector space V, we can let $[\cdot,\cdot]$ be the zero-map, i.e. $[u,v] = 0$ for all $u, v \in V$. This defines an *abelian* Lie algebra (named like this because for the matrix commutator, we have $[A,B] = 0$ if and only if A and B commute).

(ii) $\mathfrak{gl}_n = \mathrm{Mat}(n \times n)$, or, for V a vector space, $\mathfrak{gl}_V = \mathrm{End}(V)$.

(iii) $\mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid \mathrm{tr}(A) = 0\}$ (where tr denotes the *trace*). Observe that for $X, Y \in \mathfrak{sl}_n$ we have $\mathrm{tr}(XY) \neq 0$ in general, but since $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ we get $\mathrm{tr}([X,Y]) = 0$. So $[A,B] = AB - BA$ has $[\cdot,\cdot] : \Lambda^2\mathfrak{sl}_n \to \mathfrak{sl}_n$.

(iv) $\mathfrak{so}_n = \{A \in \mathfrak{gl}_n \mid A + A^T = 0\}$.

(v) $\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid JA^T J^T + A = 0\}$, where

$$J = \begin{pmatrix} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix}.$$

(vi) $\mathfrak{b}$, the upper triangular matrices in $\mathfrak{gl}_n$ ($\mathfrak{b}$ stands for Borel).

(vii) $\mathfrak{h}$, the strictly upper triangle matrices in $\mathfrak{gl}_n$.

Exercise 1.13:

(i) Check directly that $\mathfrak{gl}_n$ is a Lie algebra.

(ii) Check that the examples (iii)-(vii) are Lie subalgebras of $\mathfrak{gl}_n$.

Note that, for example, $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ is not a subalgebra of $\mathfrak{gl}_n$.

Exercise 1.14:

(i) Find algebraic groups, whose Lie algebras are those above.

(ii) Classify all Lie algebras of dimension 3 (or 2) as vector spaces. Note that the 1-dimensional Lie algebras are all abelian algebras.

Definition 1.15: A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V is a *homomorphism of Lie algebras* $\varphi\colon \mathfrak{g} \to \mathfrak{gl}_V$, i.e. a map $\varphi\colon \mathfrak{g} \to \mathrm{End}(V)$, such that

$$\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x), \quad \forall x, y \in \mathfrak{g}.$$

We say $\mathfrak{g}$ *acts* on V.

Example 1.16: If $\mathfrak{g} \subseteq \mathfrak{gl}_V$, then $\mathfrak{g}$ acts on V, so the Lie algebras from Example 1.12 act faithfully on k^n.

Definition 1.17: If $x \in \mathfrak{g}$, we define $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}(x)(y) = [x, y]$, this defines $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Lemma 1.18: ad is a representation, called the *adjoint representation.*

Proof: The identity $\mathrm{ad}[x, y] = \mathrm{ad}\, x\, \mathrm{ad}\, y - \mathrm{ad}\, y\, \mathrm{ad}\, x$ follows from skew symmetry and the Jacobi identity. □

Definition 1.19: The *center* of $\mathfrak{g}$ is $Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g} : [x, y] = 0\} = \ker \mathrm{ad}$. So $\mathfrak{g}$ has trivial center if and only if $\mathfrak{g}$ embeds via ad into $\mathfrak{gl}_{\mathfrak{g}}$.

Example 1.20: $\mathfrak{h} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ is abelian, so maps to 0 in $\mathfrak{gl}_{\mathfrak{h}}$ via ad, but $\mathfrak{h} \subseteq \mathfrak{gl}_2$ also admits a faithful representation by definition.

Theorem 1.21 (Ado): Any finite-dimensional Lie algebra over some field k is a Lie subalgebra of $\mathfrak{gl}_n$ for some n (i.e. admits a faithful finite-dimensional representation).

Example 1.22:
$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{C} \right\}$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. So a representation of $\mathfrak{sl}_2$ on k^n is a triple E, F, H of $n \times n$-matrices, such that $[E, F] = H$, $[H, E] = 2E$, $[H, F] = -2F$. This leads to the question how we can obtain such representations?

Definition 1.23: If G is an algebraic group, then an *algebraic representation* of G on a vector space V is a homomorphism of groups $\rho\colon G \to GL_V$ defined by polynomial equations in the matrix coefficients of $G \subseteq GL_n$.

Again, we can substitute $E = k[\varepsilon]/\varepsilon^2$ for K. Thus, we get a homomorphism of groups $G(E) \to GL_V(E)$. As $\rho(I) = I$, we have $\rho(I + A\varepsilon) = I + \varepsilon(\text{some function of } A)$. Call this function $\mathrm{d}\,\rho$, so $\rho(I + A\varepsilon) = I + \varepsilon\,\mathrm{d}\,\rho(A)$, which defines a map $\mathrm{d}\,\rho\colon \mathfrak{g} \to \mathfrak{gl}_V$. This gives a functor

$$\mathbf{AlgRep}_G \to \mathbf{Rep}_{\mathrm{Lie}(G)}, \; \rho \mapsto \mathrm{d}\,\rho.$$

Exercise 1.24:

(i) $\mathrm{d}\,\rho$ is the derivative of ρ, evaluated at I, i.e. $\mathrm{d}\,\rho\colon T_I G \to T_I GL_V$.

(ii) $\rho\colon G \to GL_V$ is a group homomorphism. Therefore, $\mathrm{d}\,\rho\colon \mathfrak{g} \to \mathfrak{gl}_V$ is a Lie algebra homomorphism, i.e. V is a representation of $\mathfrak{g}$.

Example 1.25: Let $G = SL_2$ and let $L(n)$ be the set of homogeneous polynomial of degree n in variables x and y. Then $L(n)$ has the basis $x^n, x^{n-1}y, \ldots, y^n$, so $\dim L(n) = n + 1$. SL_2 acts on $L(n)$ by

$$\rho_n\colon SL_2 \to \mathrm{Aut}(L(n)) = GL_{n+1}, \quad (\rho_n(g)f)(x, y) = f(ax + cy, bx + dy),$$

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2$, $f \in L(n)$. In particular, we have

ρ_0 the trivial representation,

ρ_1 the standard 2-dimensional representation on k^2,

ρ_2 here $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts, w.r.t. the above basis of $L(2)$, by the matrix

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

It is left as an exercise, to verify that SL_2 acts on $L(n)$ via ρ_n. Let us now compute representations of $\mathfrak{sl}_2$ on $L(n)$:

Remark 1.26: GL_2 acts on $\mathbb{P}^1$, and on $O(n)$, hence on $\Gamma(\mathbb{P}^1, O(n)) = S^n k^2$, and that is where these representations come from.

If we take the basis element $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $\rho_n(I+\varepsilon e)\cdot x^i y^j = x^i(\varepsilon x + y)^j = x^i y^j + \varepsilon j x^{i+1} y^{j-1}$, which says $\mathrm{d}\,\rho_n(e)\cdot x^i y^j = j x^{i+1} y^{j-1}$ if $j \geq 1$ ($\mathrm{d}\,\rho_n = 0$ if $j = 0$). This proves the first equation in the following exercise:

Exercise 1.27:

(i) For $\mathrm{d}\,\rho_n$, with ρ_n as in 1.25, we have:

$$e(x^i y^j) = \begin{cases} j x^{i+1} y^{j-1} & \text{if } j \geq 1 \\ 0 & \text{if } j = 0, \end{cases}$$
$$f(x^i y^j) = \begin{cases} i x^{i-1} y^{j+1} & \text{if } i \geq 1 \\ 0 & \text{if } i = 0, \end{cases}$$
$$h(x^i y^j) = (i-j) x^i y^j.$$

Hence $\mathrm{d}\,\rho_n(e) = x\frac{\partial}{\partial y}$, $\mathrm{d}\,\rho_n(f) = y\frac{\partial}{\partial x}$, $\mathrm{d}\,\rho_n(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

(ii) Check directly that these formulas give representations of $\mathfrak{sl}_2$ on $L(n)$.

(iii) Check directly that $L(2)$ is the adjoint representation.

(iv) Show that the formulas $e = x\frac{\partial}{\partial y}$, $f = y\frac{\partial}{\partial x}$, $h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ give an (infinite-dimensional!) representation on $k[x,y]$.

(v) Let $\operatorname{char} k = 0$. Show that $L(n)$ is an irreducible representation of $\mathfrak{sl}_2$, hence of SL_2.

Example 1.28: Let $G = \mathbb{C}^*$, then $\mathfrak{g} = \mathrm{Lie}(G) = \mathbb{C}$ with $[x,y] = 0$. A representation of $\mathfrak{g} = \mathbb{C}$ on V corresponds to $A \in \mathrm{End}(V)$, as a linear map $\rho\colon \mathbb{C} \to \operatorname{End} V$ is determined by $A = \rho(1)$. $W \subseteq V$ is a submodule if and only if $AW \subseteq W$, and ρ is isomorphic to $\rho'\colon \mathfrak{g} \to \mathrm{End}(V')$if and only if A and A' are conjugate as matrices. Hence, the classification of representations of $\mathfrak{g}$ is given by the Jordan normal forms of matrices.

As any linear transformation over $\mathbb{C}$ has an eigenvector, there is always a 1-dimensional subrepresentation of V. Therefore, V is irreducible if and only if $\dim V = 1$. Also, V is *completely decomposable* (i.e. breaks up into a direct sum of irreducible representations) if and only if A is diagonalizable.

Let $A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$, then the associated representation to A is indecomposable, but not irreducible. The invariant subspaces are $\langle e_1\rangle, \langle e_1, e_2\rangle, \ldots, \langle e_1, e_2, \ldots, e_n\rangle$, but their complements are no subspaces.

What about algebraic representations of $G = \mathbb{C}^*$? Here, the irreducible representations are $\rho_n\colon G \to GL_1 = \mathrm{Aut}(\mathbb{C})$, $z \mapsto (x \mapsto z^n x)$, $n \in \mathbb{Z}$. Moreover, every finite-dimensional representation is a direct sum of these representations.

Exercise 1.29: The functor $\rho \mapsto \mathrm{d}\,\rho$ takes ρ_n to multiplication by n in $\mathbb{C}$, and this is an irreducible representation of $\mathbb{C}$, but there are other irreducible representations, as we have seen before.

Notice that $\mathfrak{g} = (\mathbb{C}, \cdot)$ is also the Lie algebra of $G = (\mathbb{C}, +)$, so it is not surprising that its representations are different from the representations of $\mathbb{C}^*$. What *is* surprising, is the following:

Theorem 1.30 (Lie): The functor $\rho \mapsto d\rho$ is part of an equivalence of categories $\mathbf{AlgRep}_G \cong \mathbf{Rep}_{\mathrm{Lie}(G)}$ if G is a simply connected simple algebraic group. (E.g. for $G = SL_n, SO_n, SP_{2n}$).

Remark 1.31: Note that for algebraic groups, there is a different definition of simplicity. An algebraic groups is simple if it does not contain any proper nontrivial normal connected closed subgroup. Note for example, that for G a simply connected and simple algebraic group, the center does not have to be trivial, but it is finite, e.g. $Z(SL_n) = C_n$, the cyclic group with n elements.

Exercise 1.32: If G is an algebraic group, and Z is a finite central subgroup of G, then $\mathrm{Lie}(G/Z) = \mathrm{Lie}(G)$. I.e. the tangent space does not change if we identify central elements of an algebraic group.

We have now also seen that the map $\mathbf{AlgGp} \to \mathbf{LieAlg}, G \mapsto \mathrm{Lie}(G)$ is not injective.

Exercise 1.33:

(i) Let $G_n = \mathbb{C}^* \ltimes \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by $t \cdot \lambda = t^n \lambda$, i.e. $(t, \lambda)(t', \lambda') = (tt', (t')^n \lambda + \lambda'))$. Show that $G_n \cong G_m$ if and only if $n = \pm m$.

(ii) Show that $\mathrm{Lie}(G_n) \cong \mathbb{C}x + \mathbb{C}y$, $[x, y] = y$ which is independently of n.

Moreover, the map $\mathbf{AlgGp} \to \mathbf{LieAlg}$ is not surjective, Lie algebras in its image are called *algebraic Lie algebras*. This is really obvious in characteristic p. Take for example $\mathfrak{sl}_p/Z(\mathfrak{sl}_p)$. This cannot be the image of an algebraic group. In general, algebraic groups have a Jordan decomposition – every element can be written as a sum of a semisimple and a nilpotent element – and therefore the algebraic Lie algebras should have a Jordan decomposition as well.

2 Representations of $\mathfrak{sl}_2$

2.1 Classification of $\mathfrak{sl}_2$ representations

From now on, all Lie algebras and representations are over $\mathbb{C}$. For $\mathfrak{sl}_2$ we have the following basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

subject to the relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Theorem 2.1:

(i) For every $n \geq 0$, there is a unique (up to isomorphism) irreducible representation $\mathfrak{sl}_2$ of dimension $n + 1$.

(ii) Every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations (i.e. the category of finite-dimensional representations of $\mathfrak{sl}_2$ is *semisimple*, or every finite-dimensional $\mathfrak{sl}_2$ representation is *completely reducible*).

We now start proving part (i):
Let V be a representation of $\mathfrak{sl}_2$. Define the λ-*weight space* for V to be

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda v\},$$

the eigenvectors of h with eigenvalue λ.
Example: $L(n)_\lambda = \mathbb{C} \cdot x^i y^j$ if $i - j = \lambda$.
Suppose $v \in V_\lambda$. Consider ev

$$\begin{aligned} h(ev) &= (he - eh + eh)v = ([h,e] + eh)v \\ &= 2ev + e\lambda v = (2+\lambda)ev, \end{aligned}$$

so $v \in V_\lambda$ if and only if $ev \in V_{\lambda+2}$. Similarly, one shows that $v \in V_\lambda$ if and only if $fv \in V_{\lambda-2}$.

Definition 2.2: If $v \in V_\lambda \cap \ker e$, i.e. $ev = 0$ and $hv = \lambda v$, we say that v is a *highest weight vector of weight* λ.

Lemma 2.3: Let V be a representation of $\mathfrak{sl}_2$. If $v \in V$ is a highest weight vector of weight λ, then

$$W = \langle v, fv, f^2 v, \cdots \rangle$$

is an $\mathfrak{sl}_2$-invariant subspace of V, i.e. a subrepresentation.

Proof: We must show $fW \subset W$, $hW \subset W$, $eW \subset W$. Well, $fW \subset W$ is obvious. We already know that $f^k v \in V_{\lambda-2k}$, so $hW \subset W$. Show $eW \subseteq W$ by proving $ef^k v \in W$, $\forall k$. Claim:

$$e \cdot f^n v = n(\lambda - n + 1) f^{n-1} v \in W. \tag{1}$$

We prove this formula by induction on n:
The formula holds trivially for $n = 0$ since $v \in \ker e$. Assume that the formula holds for $n \geq 0$. Then

$$\begin{aligned} e \cdot f^{n+1} v &= (ef - fe + fe) f^n v \\ &= hf^n v + fef^n v \\ &= (\lambda - 2n) f^n v + n(\lambda - n + 1) f^n v, \text{ by induction hypothesis} \\ &= (n+1)(\lambda - n) f^n v. \end{aligned}$$

□

Lemma 2.4: Let V be a representation of $\mathfrak{sl}_2$ and v a highest weight vector with weight λ. If V is finite-dimensional, then $\lambda \in \mathbb{N}_0$.

Proof: The vectors $f^i v$ all lie in *different* eigenspaces of h, and hence if non-zero are linearly independent. But if V is finite-dimensional, then it must be $f^k v = 0$ for some k, so $f^{k+r} v = 0$, for all $f \geq 0$. Choose k minimal such that $f^k v = 0$. So $f^{k-1} v \neq 0$, but then

$$0 = ef^k v \overset{(1)}{=} \underbrace{k}_{\neq 0} (\lambda - k + 1) \underbrace{f^{k-1} v}_{\neq 0},$$

so $\lambda = k - 1$, i.e. $\lambda \in \mathbb{N}_0$. □

Proposition 2.5: If V is a finite-dimensional representation of $\mathfrak{sl}_2$, then there exist a highest weight vector.

Proof: Let $v \in V$ be some eigenvector for h with eigenvalue λ (exists as $\mathbb{C}$ is algebraically closed). As before, $v, ev, e^2v, ...$ are all eigenvalues for h, with respect to the distinct eigenvalues $\lambda, \lambda - 2, ...$ Hence, $v, ev, e^2v, ...$ are linearly independent unless they are zero. But V is finite-dimensional, so there exists a k s.t. $e^k v \neq 0$, but then $e^{k+1}v = e^{k+r}v = 0$, $\forall r \geq 1$. Hence, $e^k v$ is a highest weight vector with weight $\lambda + 2k$. $\square$

Corollary 2.6: If V is irreducible, then $\dim V = n + 1$, for some $n \geq 0$. We have seen that we can find a basis $v_0, v_1, ..., v_n$ with

$$
\begin{aligned}
hv_i &= (n - 2i)v_i, \\
fv_i &= \begin{cases} v_{i+1}, & \text{if } i \leq n \\ 0, & \text{if } i = n \end{cases} \\
ev_i &= i(n - i + 1)v_{i-1},
\end{aligned}
$$

i.e. there is precisely one irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$. In particular, this representation is given by $L(n)$.

This finishes the proof of part (i) of Theorem 2.1. We now prove part (ii). Notice, that the statement implies, in particular, that h acts diagonalizable on every finite-dimensional representation. First, another exercise:

Exercise 2.7: We have seen that $\mathbb{C}[x, y] = \bigoplus_{n \geq 1} L(n)$ is a representation of $\mathfrak{sl}_2$, a direct sum of irreducible representations $L(n)$, show that $x^\mu y^\lambda \in \mathbb{C}[x/y, y/x]$ is a representation of $\mathfrak{sl}_2$ for all $\lambda, \mu \in \mathbb{C}$ (using the given formulas) and describe its submodule structure.

Definition 2.8: Let V be a finite-dimensional representation of sl_2. Define

$$
\Omega := ef + fe + \frac{1}{2}h^2 \in \operatorname{End}(V). \tag{2}
$$

Ω is called the *Casimir* of $\mathfrak{sl}_2$.

Lemma 2.9: The Casimir Ω is central, i.e. $e\Omega = \Omega e$, $f\Omega = \Omega f$, $h\Omega = \Omega h$ as elements of $\operatorname{End}(V)$.

Proof: For example,

$$\begin{aligned}
e\Omega &= e(ef + fe + \frac{1}{2}h^2) \\
&= eh + 2efe + \frac{1}{2}eh^2 \\
&= eh + 2efe + \frac{1}{2}(eh - he)h + \frac{1}{2}heh \\
&= 2efe + \frac{1}{2}heh \\
&= 2efe - he + he + \frac{1}{2}heh \\
&= 2efe - (ef - fe)e + \frac{1}{2}h(he - eh) + \frac{1}{2}heh \\
&= efe + fee + \frac{1}{2}h^2e = \Omega e,
\end{aligned}$$

and similar calculations show that Ω also commutes with f and h. $\square$

Corollary 2.10: If V is an irreducible finite-dimensional representation of $\mathfrak{sl}_2$, then Ω acts on it by a scalar.

Proof: Since Ω is central, $\rho_\Omega \colon V \to V,\ v \mapsto \Omega v$ defines an $\mathfrak{sl}_2$-linear map, then Schur's Lemma gives that $\rho_\Omega = \lambda \mathrm{Id}_V$, for some $\lambda \in \mathbb{C}$. $\square$

Lemma 2.11: Let $L(n)$ denote the irreducible representation with highest weight vector v, of weight n, then Ω acts on $L(n)$ by $\frac{1}{2}n(n+2)$.

Proof: We have $ev = 0$, $hv = nv$, $\Omega = (\frac{1}{2}h^2 + h) + 2fe$, so $\Omega v = \frac{1}{2}n(n+2)v$ by Schur's Lemma. Notice, that Schur's Lemma is actually not needed here. One can simply apply that $\Omega f^i v = f^i \Omega v$, and $\{f^i v \mid i \in \mathbb{N}\}$ span $L(n)$. $\square$

Observe, if $L(n)$ and $L(m)$ are two irreducible finite-dimensional representations of $\mathfrak{sl}_2$, and Ω acts on them by the same scalar, then $n = m$.

Proof: Ω acts by $\frac{1}{2}n^2 + 2 = \frac{1}{2}m^2 + m$, but $f(x) = \frac{1}{2}x^2 + x$ is a strictly increasing function for $x > -1$. $\square$

Let V be any finite-dimensional representation of $\mathfrak{sl}_2$, set

$$V^\lambda := \langle v \in V \mid (\Omega - \lambda)^{\dim V^\lambda} v = 0 \rangle,$$

the *generalized eigenspace* of Ω with eigenvalue λ. Using *Jordan decomposition*, we can decompose $V = \bigoplus_\lambda V^\lambda$, and write Ω in *Jordan normal form*

$$\Omega = \begin{pmatrix} \lambda_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && \lambda_1 &&&&&& \\ &&& \lambda_2 & 1 &&&& \\ &&&& \ddots & \ddots &&& \\ &&&&& \ddots & 1 && \\ &&&&&& \lambda_2 && \\ &&&&&&& \ddots & \\ &&&&&&&& \lambda_n \end{pmatrix},$$

for some generalized eigenvalues $\lambda_1, ...\lambda_n \in \mathbb{C}$. The Jordan blocks correspond to generalized eigenspaces V^λ.

Claim 2.12: Each V^λ is a subrepresentation of $\mathfrak{sl}_2$.

Proof: Let $x \in \mathfrak{sl}_2$, $v \in V^\lambda$, then

$$\begin{aligned}(\Omega - \lambda)^{\dim V} xv &= x(\Omega - \lambda)^{\dim V} v \\ &= x0 = 0,\end{aligned}$$

so $xv \in V^\lambda$ also. □

If $V^\lambda \neq 0$, then $\lambda = \frac{1}{2}n^2 + n$ for a unique $n \geq 0$, and we can show, that each V^λ is 'glued together' from copies of the representation $L(n)$. This can be formulated more precisely using composition series.

Definition 2.13: Let W be a finite-dimensional $\mathfrak{g}$-module, for a Lie algebra $\mathfrak{g}$. A *composition series* for W is a sequence of submodules

$$0 = W_0 < W_1 < ... < W_r = W,$$

such that each quotient W_i/W_{i-1} is an irreducible module.

Example 2.14:

(i) If $\mathfrak{g} = \mathbb{C}$, $W = \mathbb{C}^r$, where $1 \in \mathbb{C}$ acts as the matrix $\begin{pmatrix} 0 & 1 && \\ & \ddots & \ddots & \\ && \ddots & 1 \\ &&& 0 \end{pmatrix}$, then there is a *unique* composition series

$$0 < \langle e_1 \rangle < \langle e_1, e_2 \rangle < ... < \langle e_1, e_2, ..., e_r \rangle,$$

and the subquotients are all $\mathbb{C}$ (the trivial module).

(ii) If $\mathfrak{g} = \mathbb{C}$, $W = \mathbb{C}^r$, $1 \in \mathbb{C}$ acts as 0, then any chain

$$0 < W_1 < ... < W_r = W,$$

with $\dim W_i = i$, is a composition series and again, the subquotients are $\mathbb{C}$.

Claim 2.15: Composition series exist for any finite-dimensional $\mathfrak{g}$-module W.

Proof: By induction on $\dim W$. Take any irreducible submodule $W_1 < W$, then W/W_1 is of smaller dimension, so has a composition series

$$0 < \overline{W}_2 < \overline{W}_3 < ... < \overline{W}_r = W/W_1,$$

by induction, then

$$0 < W_1 < W_1 + \overline{W}_2 < W_1 + \overline{W}_3 < ... < W_1 + \overline{W}_{r-1} < W_r = W,$$

is a composition series of W. $\square$

Lemma 2.16: Let $V^\lambda \neq 0$, then $\lambda = \frac{1}{2}n^2 + n$ for a unique n, and V^λ has a composition series, such that all quotients are isomorphic to $L(n)$.

Proof: Let W be an irreducible submodule of V^λ, Ω still acts on W by λ, but W is $L(n)$, for some unique $n \geq 0$, and so $\lambda = \frac{1}{2}n^2 + n$. Now consider V^λ/W. If this space is non-zero, Ω still acts on V^λ/W with only one generalized eigenvalue λ. We can repeat this procedure as long as the quotient is non-zero. This shows that V^λ has a composition series with $L(n)$ as the only module which appears as a quotient, i.e. $W_i/W_{i-1} = L(n)$, for all i. $\square$

Corollary 2.17: h acts on V^λ with (generalized) eigenvalues in $\{n, n-2, ..., 2-n, -n\}$.

Proof: If h acts on W, $W' \leq W$ a subspace s.t. $hW' \leq W'$, then

$$\{\text{gen. eigenvalues of } h \text{ on } W\} = \{\text{gen. ev. of } h \text{ on } W'\} \cup \{\text{gen. ev. of } h \text{ on } W/W'\}. \tag{3}$$

If we apply this to V^λ, we obtain

$$\{\text{generalized eigenvalues of } h \text{ on } V^\lambda\} = \{\text{eigenvalues of } h \text{ on } L(n)\}.$$ $\square$

Lemma 2.16 says that the action of h on V^λ has the following form:

$$\begin{pmatrix} L(n) & * & * & * \\ 0 & L(n) & * & * \\ & \ddots & \ddots & * \\ & & 0 & L(n) \end{pmatrix}.$$

Using Jordan normal form, $L(n)$ has diagonal form with eigenvalues $\{n, n-2, ..., -n+2, -n\}$ on the diagonal. So these are the only generalized eigenvalues of V^λ, i.e. $(V^\lambda)_m = 0$ if $m \notin \{n, n-2, ..., -n+2, -n\}$. Further, h acts on $\ker(e\colon V^\lambda \to V^\lambda)$ with only one generalized eigenvalue, namely n, i.e. if $x \in \ker e$, then $(h-n)^{\dim V^\lambda} \cdot x = 0$. To see this, apply (3) to the composition series given by $\overline{W}_i = W_i \cap \ker e$, if the W_i come from the composition series of V^λ, developed in Lemma 2.16.

Lemma 2.18: For the endomorphisms given by elements of $\mathfrak{sl}_2$ acting on V^λ, the following identities hold:

(i) $hf^n = f^n(h-2n)$

(ii) $ef^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$

Proof:

(i) By induction on n:
For $n=1$, we have $hf = hf - fh + fh = -2f + fh = f(h-2)$.
Assume, the formula holds for $n \geq 1$. Then

$$\begin{aligned} hf^{n+1} &= f^n(h-2n)f = f^n hf - 2nf^{n+1} \\ &= -2f^{n+1} + f^{n+1}h - 2nf^{n+1} \\ &= f^{n+1}(h - 2(n+1)) \end{aligned}$$

(ii) By induction on n:
For $n=0$, observe $ef = ef - fe + fe = h + fe$.
Assume, that the formula holds for $n \geq 0$, then

$$\begin{aligned} ef^{n+2} &= (f^{n+1}e + (n+1)f^n(h-n))f \\ &= f^{n+1}ef + (n+1)f^n hf - n(n+1)f^{n+1} \\ &= f^{n+1}h + f^{n+2}e - 2(n+1)f^{n+1} + (n+1)f^{n+1}h - n(n+1)f^{n+1} \\ &= f^{n+2}e + (n+2)f^{n+1}(h-(n+1)) \end{aligned}$$

□

Proposition 2.19: h acts diagonalizable on $\ker(e\colon V^\lambda \to V^\lambda)$, i.e.

$$\ker e = (V^\lambda)_n = \{x \in V^\lambda \mid hx = nx\}.$$

Proof: "$\supseteq$": If $hx = nx$, then $ex \in (V^\lambda)_{n+2} = 0$, so $x \in \ker e$.
"$\subseteq$": Let $x \in \ker e$. We showed in Corollary 2.17 that in this case

$$(h-n)^{\dim V^\lambda} x = 0. \tag{4}$$

Now, by part (i) of Lemma 2.18:

$$(h-n+2k)^{\dim V^\lambda} f^k x = 0,$$

i.e. $f^k x$ lies in the generalized eigenspace for h with eigenvalue $n-2k$ $(*)$. On the other hand, if $y \in \ker e$, and $y \neq 0$, then $f^k y \neq 0$ $(**)$. To prove this, let

$$0 = W_0 < W_1 < ... < W_r = V^\lambda$$

be a composition series for V^λ. There exists an i s.t. $y \notin W_i$, but $y \in W_{i-1}$, put $\overline{y} = y + W_{i-1}$. Note that $\overline{y} \neq 0 \in W_i/W_{i-1} \cong L(n)$. Then $\overline{y}$ is a highest weight vector for $L(n)$, so $f^n\overline{y} \neq 0$ in $L(n)$, so $f^n y \neq 0$ in V^λ. Now, $f^{n+1}y$ lies in the generalized eigenspace for h with eigenvalue $-n-2$, by $(*)$, but this is the zero space. Hence, $f^{n+1}y = 0$. Now, by applying Lemma 2.18, we can conclude

$$0 = ef^{n+1}y = (n+1)f^n(h-n)y + f^{n+1}\underbrace{ey}_{=0},$$

so $f^n(h-n)y = hf^n y = 0$, but if $(h-n)y \neq 0$, this would contradict $(**)$, so $hy = ny$. □

Now we can finish our proof of Theorem 2.1. We can now choose a basis $w_1, ..., w_k$ of $\ker(e\colon V^\lambda \to V^\lambda)$ s.t. $hw_i = nw_i$ and $ew_i = 0$, by Proposition 2.19 (i.e. $\ker e = (\ker e)_n$. This gives a direct sum composition of V^λ, using the basis $w_1, fw_1, ..., f^n w_1, ..., w_k, fw_k, ..., f^n w_k$. h acts diagonalizable on V^λ with respect to this basis, and hence on the whole of V (using Jordan decomposition). To convince ourselves, that this is true, consider

$$\begin{aligned} hf^k w_i &= f^k(h-2k)w_i \\ &= f^k(n-2k)w_i \\ &= (n-2k)f^k w_i, \end{aligned}$$

i.e. $f^k w_i \in (V^\lambda)_{n-2k}$, and h acts diagonalizable on the whole of V^λ. This concludes the proof of Theorem 2.1.

Exercise 2.20 (fun!): Show that, if $\mathrm{char} k = p$, then

(i) irreducible highest weight representations of $\mathfrak{sl}_2(\mathbb{F}_p)$ are parametrized by $n \in N$, and

(ii) arbitrary finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{F}_p)$ do not need to break up into a direct sum of irreducibles.

2.2 Consequences

Let V, W be representations of a Lie algebra $\mathfrak{g}$.

Claim 2.21: The map $\mathfrak{g} \to \mathrm{End}(V \otimes W) = \mathrm{End}(V) \otimes \mathrm{End}(W)$ given by $x \mapsto x \otimes 1 + 1 \otimes x$ is a homomorphism of Lie algebras.

Proof: This map is obviously linear. To see that it is a Lie algebra morphism, consider

$$\begin{aligned} [x\otimes 1 + 1\otimes x, y\otimes 1 + 1 \otimes y] &= (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (y\otimes 1 + 1\otimes y)(x\otimes 1 + 1\otimes x) \\ &= xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy - (yx\otimes 1 + y\otimes x + x\otimes y + 1\otimes yx) \\ &= (xy - yx)\otimes 1 + 1\otimes (xy - yx) \qquad \square \end{aligned}$$

Remark 2.22: This comes from the group homomorphism $G \to G \times G$, $g \mapsto (g, g)$ by differentiating.

Corollary 2.23: If V, W are representations of $\mathfrak{g}$, so is $V \otimes W$.

Remember that if A is an algebra, V, W representations of A, then $V \otimes W$ is a representation of $A \otimes A$. To make it a representation of A, we need an algebra homomorphism $A \to A \otimes A$ (such a map is called *coproduct* of a *Hopf algebra*).

Now, take $\mathfrak{g} = \mathfrak{sl}_2$. This gives rise to the question how $L(n) \otimes L(m)$ breaks up into a direct sum of irreducibles $L(i)$ (using Theorem 2.1). One method to answer this question is to find all the highest weight vectors.

Exercise 2.24: Find all highest weight vectors in $L(1) \otimes L(m), ..., L(n) \otimes L(m)$.
Easy start: $L(n) \otimes L(m)$. Write v_n for the highest weight vector in $L(n)$, we claim that $v_n \otimes v_m$

is a highest weight vector in $L(n) \otimes L(m)$. To prove this, consider

$$\begin{aligned} h \cdot (v_n \otimes v_m) &= (hv_n) \otimes v_m + v_n \otimes (hv_m) \\ &= (n+m) v_n \otimes v_m, \qquad \text{and} \\ e \cdot (v_n \otimes v_m) &= (ev_n) \otimes v_m + v_n \otimes (ev_m) = 0. \end{aligned}$$

From this, we can conclude that $L(n) \otimes L(m) = L(n+m) + X$, but since

$$\begin{aligned} (n+1)(m+1) &= \dim L(n) \otimes L(m) \\ &= \dim L(n+m) + \dim X \\ &= n + m + (nm+1), \end{aligned}$$

there is still "lots of stuff" remaining, if we quotient out by the submodule $L(n+m)$. One strategy to find this "other stuff" is to write down explicit formulas for all the other highest weight vectors. These are complicated, but mildly interesting.

However, to determine the summands of $L(n) \otimes L(m)$ we do not have to do this.

Definition 2.25: Let V be a finite-dimensional representation of $\mathfrak{sl}_2$. The *character* of V is defined as

$$\operatorname{ch} V = \sum_{n \in \mathbb{Z}} \dim V_n z^n \in \mathbb{N}[z, z^{-1}].$$

Lemma 2.26: Let V, W be $\mathfrak{sl}_2$-representations, then

(i) $\operatorname{ch} V|_{z=1} = \dim V$,

(ii) $\operatorname{ch} L(n) = z^n + z^{n-2} + ... + z^{-n+2} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$, sometimes denoted as $[n+1]_z$,

(iii) $\operatorname{ch} V = \operatorname{ch} W \iff V \cong W$,

(iv) $\operatorname{ch} V \otimes W = chV \cdot chW$.

Proof:

(i) h acts diagonalizable with all its eigenvalues integers, i.e. $V = \bigoplus_{n \in \mathbb{Z}} V_n$ by Theorem 2.1.

(ii) Follows from Theorem 2.1.

(iii) The characters $\operatorname{ch} L(0) = 1, \operatorname{ch} L(1) = z + z^{-1}, \operatorname{ch} L(2) = z^2 + 1 + z^{-2}, ...$ form a basis of $\mathbb{Z}[z, z^{-1}]^{\mathbb{Z}/2}$, the symmetric Laurent polynomials. Clearly, they are linearly independent and span this space (by inspection). On the other hand, by part (ii) of Theorem 2.1 (complete reducibility), we have

$$V \cong \bigoplus_{n \geq 0} a_n L(n), \quad W \cong \bigoplus_{n \geq 0} b_n L(n),$$

and $V \cong W$ if and only if $a_n = b_n$ for all $n \in \mathbb{N}$. But now, as $\{\operatorname{ch} L(n)\}_n$ forms a basis, $\operatorname{ch} V = \sum_{n \geq 0} a_n \operatorname{ch} L(n)$ determines a_n.

(iv) Since for $v_n \in V_n,\ v_m \in V_m$, we have $h(v_n \otimes v_m) = (n+m)v_n \otimes v_m$, we see that $V_n \otimes V_m \subseteq (V \otimes W)_{n+m}$, so

$$\begin{aligned} (V \otimes W)_p &= \sum_{\substack{n,m \\ n+m=p}} V_n \otimes V_m, \\ \Rightarrow\ \dim(V \otimes W)_p &= \sum_{\substack{n,m \\ n+m=p}} (\dim V_n)(\dim V_m) \\ \Rightarrow\ \operatorname{ch} V \otimes W &= \sum_{p \in \mathbb{Z}} \sum_{\substack{n,m \\ n+m=p}} (\dim V_n)(\dim V_m) z^p \\ &= (\operatorname{ch} V)(\operatorname{ch} W), \end{aligned}$$

since this is how we multiply polynomials. □

Example 2.27: Decompose $L(1) \otimes L(3)$.

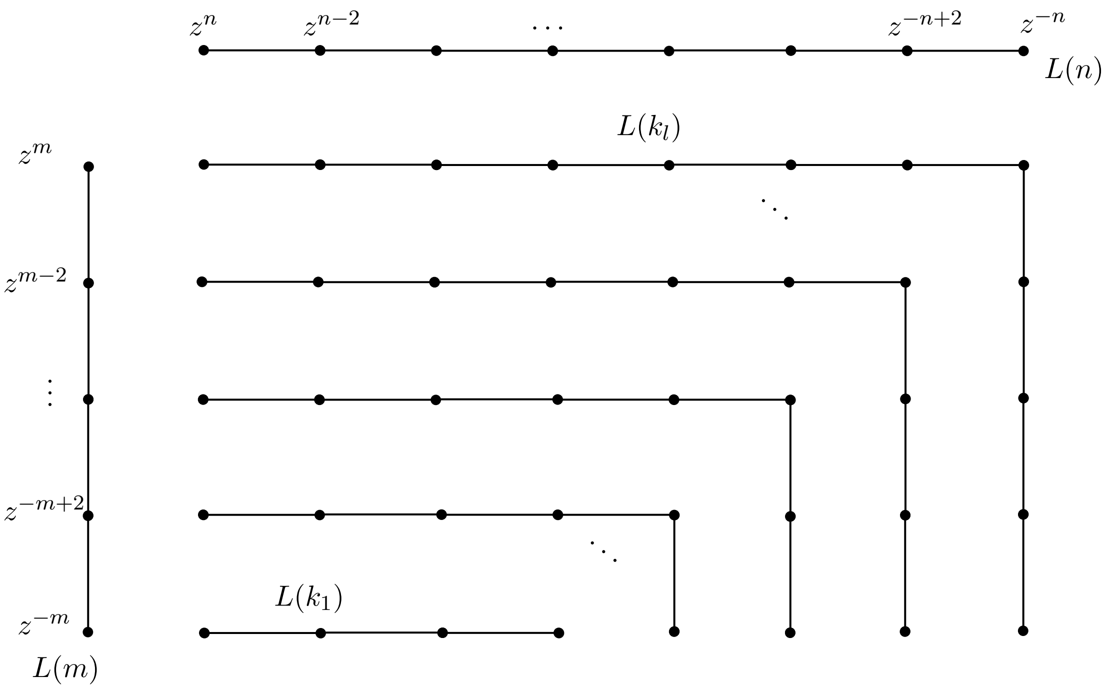
$$\begin{aligned} \operatorname{ch} L(1) \otimes L(3) &= (z + z^{-1})(z^3 + z + z^{-1} + z^{-3}) \\ &= (z^4 + z^2 + 1 + z^{-2} + z^{-4}) + (z^2 + 1 + z^{-2}) \end{aligned}$$

hence $L(1) \otimes L(3) \cong L(4) \oplus L(2)$.

We can use the *Clebsch-Gordon-rule*

$$L(n) \otimes L(m) = \bigoplus_{\substack{k=|n-m| \\ k \equiv n-m \pmod 2}}^{n+m} L(k). \tag{5}$$

Without giving a formal prove, the formula can be verified by drawing diagrams:

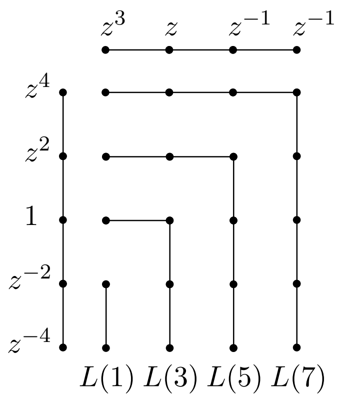


Here, $L(k_1), ..., L(k_l)$ are the components of the direct sum composition of $L(n) \otimes L(m)$, and $k_1, ..., k_l$ refer to the respective length of the lines in the inner square of the diagram.

Example 2.28: Compute $L(3)\otimes L(4)$, $\operatorname{ch} L(3)\otimes L(4) = (z^3+z+z^{-1}+z^{-3})(z^4+z^2+z^{-2}+z^{-4})$. Since z^7 appears as the highest coefficient in the product of the characters, $L(7)$ appears in the decomposition. Subtracting $\operatorname{ch} L(7)$, the highest coefficient is z^5 and thus $L(5)$ appears as an summand. Continuing in a similar manner, we conclude

$$L(3) \otimes L(4) = L(7) \oplus L(5) \oplus L(3) \oplus L(1).$$

This can also be seen by considering the diagram



Here $L(7)$ refers to the line of length 7, $L(5)$ refers to the line of length 5,...

In the following chapters we will look at other Lie algebras including $\mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$. We will

- see that the categories of representations are semi-simple,
- parametrize irreducible representations,
- compute the character of the irreducibles and their dimensions,
- see how to decompose $\otimes$ using picture *crystals*.

In order to do this, we need

- linear algebra characterizations of such Lie algebras, and
- the structure theory of *roots and weights*.

3 Structure and Classification of Simple Lie Algebras

3.1 Linear algebra preliminaries

Definition 3.1:

(i) A Lie algebra $\mathfrak{g}$ is *simple* if the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$ and $\dim \mathfrak{g} > 1$ ($\iff \mathfrak{g}$ is non-abelian). $\mathfrak{g}$ is *semi-simple* if it is a direct sum of simple Lie algebras.

(ii) $[\mathfrak{g}, \mathfrak{g}]$ is the span of $\{[X, Y] : X, Y \in \mathfrak{g}\}$, the *derived algebra* of $\mathfrak{g}$.

(iii) The *central series* of $\mathfrak{g}$ is defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}]$, i.e

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \supseteq \ldots$$

The *derived series* is: $\mathfrak{g} < (0) = \mathfrak{g}$ and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$, i.e.

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \ldots$$

(iv) $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some $n > 0$, *solvable* if $\mathfrak{g}^{(n)} = 0$ for some $n > 0$.

Remark 3.2: $\mathfrak{g}$ nilpotent implies $\mathfrak{g}$ solvable since we always have $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$.

Exercise 3.3: Show $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian.

Solution: $[\mathfrak{g}, \mathfrak{g}]$ is an ideal since $[g, [a, b]] \in [\mathfrak{g}, \mathfrak{g}]$, for all $g, a, b \in \mathfrak{g}$. To see that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, consider

$$\begin{aligned} [a + [\mathfrak{g}, \mathfrak{g}], b + [\mathfrak{g}, \mathfrak{g}]] &= [a, b] + [\mathfrak{g}, \mathfrak{g}] \\ &= [\mathfrak{g}, \mathfrak{g}]. \end{aligned}$$

□

Example 3.4: The Lie algebra $\mathfrak{h}$ of strictly upper triangular matrices is nilpotent, the Lie algebra $\mathfrak{b}$ of upper triangular matrices is solvable.

Exercise 3.5:

(i) Compute the derived and the central series of $\mathfrak{h}$ and $\mathfrak{b}$ and check the above claim.

(ii) Compute the center of $\mathfrak{h}$ and $\mathfrak{b}$.

Let W be a *symplectic* vector space, i.e. a vector space with an *inner product* $\langle\ ,\ \rangle$ (that is an non-degenerate antisymmetric bilinear form). For example, take L to be any vector space, set $W = L + L^*$, and define $\langle L, L\rangle = \langle L^*, L^*\rangle = 0$, $\langle v,\ v^*\rangle = v^*(v) = -\langle v^*, v\rangle$, $\forall v \in L, v^* \in L^*$.

Exercise 3.6: Define the Heisenberg Lie algebra $\mathscr{H}_W := W \oplus \mathbb{C}c$ as a vector space, $[w, w'] = \langle w, w'\rangle c$, for $w, w' \in W$, and $[c, w] = 0$. Show that $\mathscr{H}_W$ is a Lie algebra that is nilpotent.

Solution. $[\ ,\ ]$ is bilinear and skew-symmetric since $\langle\ ,\ \rangle$ is. Note that $[\ ,\ ] \subseteq \mathbb{C}c$. Thus, the Jacobi identity follows immediately from $[c, w] = 0$. Because of the same property, we have that $\mathscr{H}_W^2 = [[\mathscr{H}_W, \mathscr{H}_W], \mathscr{H}_W] = [\langle W, W\rangle c, \mathscr{H}_W] = 0$. This proves that $\mathscr{H}_W$ is nilpotent, and thus also solvable. □

Example 3.7: Let $L = \mathbb{C}$, then $\mathscr{H}_W = \mathbb{C}_p + \mathbb{C}_q + \mathbb{C}_c$, $[p, q] = c, [c, p] = [c, q] = 0$. Show that this has a representation on $\mathbb{C}[x]$ by $q \mapsto x$, $p \mapsto \frac{\partial}{\partial x}$, $c \mapsto 1$.

Solution: We need to check that the defined map preserves the relations $[p, q] = c, [c, p] = 0, [c, q] = 0$:

$$\begin{aligned} \left[\frac{\partial}{\partial x}, x\right] x^n &= \left(\frac{\partial}{\partial x} x - x\frac{\partial}{\partial x}\right) x^n \\ &= (n+1)x^n - nx^n = 1 \cdot x^n, \\ \left[\frac{\partial}{\partial x}, 1\right] x^n &= \left(\frac{\partial}{\partial x} 1 - 1\frac{\partial}{\partial x}\right) x^n = 0, \\ [x, 1]x^n &= (x - x)x^n = 0. \end{aligned}$$

This shows that the defined map is a representation of $\mathscr{H}_W$. □

Proposition 3.8:

(i) Subalgebras and quotient algebras of solvable (resp. nilpotent) Lie algebras are solvable (resp. nilpotent).

(ii) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ an ideal. Then we have $\mathfrak{g}$ solvable $\iff$ $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. (So solvable Lie algebras are built out of abelian Lie algebras, it exists a refinement of derived series s.t. the subquotients are 1-dimensional).

(iii) $\mathfrak{g}$ is nilpotent if and only if the center $Z(\mathfrak{g}) \neq 0$ and $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. (Indeed: if $\mathfrak{g}$ nilpotent $\mathfrak{g} \supset \mathfrak{g}^1 \supset \ldots \supset \mathfrak{g}^{n-1} \supset \mathfrak{g}^n = 0$. But $0 = \mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}] \Rightarrow \mathfrak{g}^{n-1}$ lies in the center of $\mathfrak{g}$.)

(iv) In particular, $\mathfrak{g}$ is nilpotent $\iff$ $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{gl}(G)$ is nilpotent (as $0 \to Z(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathrm{ad}(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g}) \to 0$ is an exact sequence).

Theorem 3.9 (Lie's Theorem): Let $\mathfrak{g} \subseteq \mathfrak{gl}_V$ be a solvable Lie algebra over an algebraic closed field k with $\operatorname{char} k = 0$. Then there exists a basis $v_1, \ldots, v_n$ of V such that w.r.t. this basis the matrices of all elements of $\mathfrak{g}$ are upper triangular, i.e. $\mathfrak{g} \subseteq \mathfrak{b}_V$.

Equivalently, there exists a $\lambda\colon \mathfrak{g} \to k$ linear and $v \in V$ st. $xv = \lambda(x)v$ for all $x \in \mathfrak{g}$ (that is v is a *common eigenvector* for $\mathfrak{g}$, i.e. a one-dimensional subrepresentation of V).

Exercise 3.10:

(i) Show these are equivalent.

(ii) Show it is necessary that $K = \overline{K}$ and $\operatorname{char} K = 0$. For example, take $\mathfrak{g} = \mathscr{H}_W = \langle p, q, c \rangle, \operatorname{char} K = p$ and show $K[x]/K$ is an irreducible representation of $\mathfrak{g}$, contradicting Lie.

Solution:

(i) Assume first statement of Lie's Theorem. Fix basis $v_1, ..., v_n$ such that all elements of $\mathfrak{g}$ act as an upper triangle matrix. Note that for any $h \in \mathfrak{g}$ we have $hv_1 = \lambda_h$ for some $\lambda_h \in K$. Define common eigenvector by $h \mapsto \lambda_h$.
Conversely, if we have a one-dimensional subrepresentation V_1 of V, take $0 \neq v_1 \in V_1$ as first basis vector. Assume, we have found $v_1, ..., v_k$ basis vectors such that every $h \in \mathfrak{g}$ acts as an upper triangular matrix on $W_k := \langle v_1, ..., v_k \rangle$. Then V/W_k is still solvable, and we again find a one-dimensional subrepresentation V_{k+1}. Take $0 \neq v_{k+1} \in V_{k+1}$. Then $\langle hv_{k+1} \rangle \cap W_k = 0$ and therefore h acts as an upper triangular matrix on $W_{k+1} := \langle W_k, v_{k+1} \rangle$. Thus, the claim follows by induction. □

Corollary 3.11: Let $\operatorname{char} k = 0$, $\mathfrak{g}$ a solvable finite-dimensional Lie algebra, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof: Apply Lie's theorem to the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Then – w.r.t. some basis – $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{b}$, but $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{h}$, so $[\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}]$ is nilpotent. Moreover, $[\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}] = \mathrm{ad}[\mathfrak{g}, \mathfrak{g}]$, so $[\mathfrak{g}, \mathfrak{g}]$ has to be nilpotent (by property (iv) of Proposition 3.8). □

Exercise 3.12: Find a counterexample to the previous corollary for $\operatorname{char} k = p$.

We call a endomorphism $\phi\colon V \to V$ *nilpotent* if all its eigenvalues are zero. Or, equivalently, if $\phi^n = 0$ for some $n \geq 0$.

Theorem 3.13 (Engel's Theorem): Let k be an arbitrary field. $\mathfrak{g}$ is a nilpotent Lie algebra if and only if $\mathrm{ad}(\mathfrak{g})$ consists of nilpotent endomorphisms of $\mathfrak{g}$. Or, equivalently, if (V, π) is a finite-dimensional representation of $\mathfrak{g}$ such that $\pi(x)\colon V \to V$ is a nilpotent endomorphism for all $x \in \mathfrak{g}$, then there exists $0 \neq v \in V$ st. $\pi(x)v = 0$ for all $x \in \mathfrak{g}$ (i.e. V has a trivial subrepresentation). This again, implies the existence of a basis such that all matrices $\pi(x)$ are strictly upper triangular.

Exercise 3.14: Show that the two formulations of Engel's Theorem are equivalent.

Solution:

$\Leftarrow$: Claim: For V exists a series

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

s.t. $\dim V_i = i$ and $\mathfrak{g}V_i \subset V_{i-1}$. Note that from the claim it follows that we can find a basis for V s.t. g acts as an strictly upper triangular matrix, for all $g \in \mathfrak{g}$. Setting $V = \mathrm{ad}(\mathfrak{g})$ gives that $\mathrm{ad}\,\mathfrak{g}$ is nilpotent and therefore $\mathfrak{g}$ is nilpotent.
Proof of the claim by induction on $n = \dim V$: If $\dim V = 1$, we have that $V = kv$ is abelian. In this case, set $V_1 = V$. Now, let $\dim V > 1$, then we find an one-dimensional subrepresentation V_1 of V. Consider $V' = V/V_1$ with canonical projection π, then $\dim V' < \dim V$ and we can apply the induction hypothesis to obtain a series

$$0 = V'_0 < V'_1 < \ldots < V'_{n-1} = V'$$

with the claimed properties. Now, the series defined by $V_i = \pi^{-1}(V'_{i-1}),\ i = 1, \ldots, n-1$ and $V_n = V$ gives the claim as $\mathfrak{g}(V_i) = \mathfrak{g}\pi^{-1}(V'_{i-1}) \subset \pi^{-1}(V'_{i-2}) = V_{i-1}$.

$\Rightarrow$: First, show that if (V, π) consists of nilpotent endomorphism, then also ad does. Assume $\pi(x)^n = 0,\ x \in \mathfrak{g}$. □

Definition 3.15: A symmetric bilinear form $(\cdot,\cdot)\colon \mathfrak{g}\times\mathfrak{g} \to k$ is *invariant* if $([x,y],z) = (x,[y,z])$ for all $x,y,z \in \mathfrak{g}$.

Exercise 3.16: If $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal, $(\cdot,\cdot)$ an invariant form on $\mathfrak{g}$, then $\mathfrak{a}^\perp = \{x \in \mathfrak{g} : (x, \mathfrak{a}) = 0\}$ is an ideal.

Solution: To check: for $a \in \mathfrak{a}^\perp$, $[a,\mathfrak{g}] \subseteq \mathfrak{a}^\perp$, i.e. $([a,\mathfrak{g}],\mathfrak{a}) = 0$. But since $(\cdot,\cdot)$ is invariant, we have $([a,\mathfrak{g}],\mathfrak{a}) = (a,[\mathfrak{g},\mathfrak{a}]) = (a,\mathfrak{a}) = 0$, where we use that $\mathfrak{a}$ is an ideal. □

Definition 3.17: Let V be a representation of $\mathfrak{g}$ via $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, then we define

$$(x,y)_V := \mathrm{tr}(\rho(x)\rho(y)\colon V \to V),$$

the *trace form* of V.

Exercise 3.18: Check that ρ is a representation implies that $(\cdot,\cdot)_V$ is symmetric, bilinear and invariant.

Solution: The trace form is symmetric as $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Bilinearity follows from linearity of ρ and tr. Check that the trace form is invariant:

$$\begin{aligned}\operatorname{tr}(\rho[x,y]\rho(z)) &= \operatorname{tr}(\rho(x)\rho(y)\rho(z) - \rho(y)\rho(x)\rho(z)) \\ &= \operatorname{tr}(\rho(x)\rho(y)\rho(z)) - \operatorname{tr}(\rho(x)\rho(z)\rho(y)) \\ &= \operatorname{tr}(\rho(x)\rho[y,z]).\end{aligned}$$ □

Example 3.19: Define $(\cdot,\cdot)_{\mathrm{ad}}$ the *killing form*, to be the trace form attached to the adjoint representation, i.e. $(x,y)_{\mathrm{ad}} = \operatorname{tr}(\operatorname{ad} x \cdot \operatorname{ad} y \colon \mathfrak{g} \to \mathfrak{g})$.

Theorem 3.20 (Cartan's Criterion): Let $\mathfrak{g} \subseteq \mathfrak{gl}_V$, $\operatorname{char} k = 0$, then $\mathfrak{g}$ is solvable if and only if for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g},\mathfrak{g}]$ we have $(x,y)_V = 0$, i.e. $[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}^{\perp}$.

Exercise 3.21: Observe that Lie's theorem implies Cartan's criterion immediately. If $\mathfrak{g}$ is solvable and non-abelian, then all trace forms are degenerate.

Solution: If $\mathfrak{g}$ is solvable, use Lie's theorem to find a basis of a representation (V,ρ) such that $\rho\mathfrak{g} \subseteq \mathfrak{b}$. Then $[\rho\mathfrak{g},\rho\mathfrak{g}] \subseteq \mathfrak{h}$, the set of upper triangular matrices with only zeros on the diagonal. Now it is clear, that $\operatorname{tr}(\rho[x,y]\rho z)$ is 0. □

Corollary 3.22: A Lie algebra $\mathfrak{g}$ is solvable if and only if $(\mathfrak{g},[\mathfrak{g},\mathfrak{g}])_{\mathrm{ad}} = 0$.

Proof: $\Rightarrow$: is Lie's Theorem.
$\Leftarrow$: Cartan's criterion gives that $\operatorname{ad}(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$ is solvable. But the center is abelian and so it is always solvable. Therefore $\mathfrak{g}$ is solvable, too. □

Warning: Not every invariant form is a trace form.

Exercise 3.23: Let $\tilde{\mathscr{H}} = \mathbb{C}\langle p,q,c,d\rangle$ with $[c,\tilde{\mathscr{H}}] = 0, [p,q] = c, [d,p] = p, [d,q] = -q$. Construct a non-degenerate invariant form on $\tilde{\mathscr{H}}$. Show that $\tilde{\mathscr{H}}$ is solvable. Extend the representation of $\mathbb{C}\langle c,p,q\rangle$ on $k[x]$ (given in 3.7) to a representation of $\tilde{\mathscr{H}}$.

3.2 Structure of semisimple Lie algebras

Definition 3.24: Let $R(\mathfrak{g})$ denote the maximal solvable ideal in $\mathfrak{g}$, the *radical* of $\mathfrak{g}$.

Exercise 3.25:

(i) Show that the sum of solvable ideals is solvable, i.e. $R(\mathfrak{g})$ is the sum of all solvable ideals.

(ii) Show $R(\mathfrak{g}/R(\mathfrak{g})) = 0$.

Definition 3.26: A *derivation* is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying $D[x,y] = [Dx,y] + [x,Dy]$ (e.g. $\operatorname{ad}(x)$ is a derivation (follows from Jacobi identity and skew symmetry of $[\cdot,\cdot]$)). A derivation of the form $\operatorname{ad}(x)$ is called *inner*.

Theorem 3.27: Let $\operatorname{char} k = 0$, then the following are equivalent:

(i) $\mathfrak{g}$ is semisimple

(ii) $R(\mathfrak{g}) = 0$

(iii) The Killing form $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate (*killing criterion*).

Moreover, if $\mathfrak{g}$ is semisimple, then every derivation $D : \mathfrak{g} \to \mathfrak{g}$ is inner. (But not conversely, i.e. this does not characterize semisimple Lie algebras.)

Proof: First notice that $R(\mathfrak{g}) = 0 \Leftrightarrow \mathfrak{g}$ has no non-zero abelian ideal. "$\Rightarrow$" clear since abelian ideals are always solvable; "$\Leftarrow$": if some ideal $\mathfrak{p} \subseteq \mathfrak{g}$ is solvable, then the last term of its derivated series is abelian.
Therefore, (i)$\Rightarrow$(ii) clear (if $\mathfrak{g}$ is semisimple, it does not have non-zero abelian ideals).
(iii)$\Rightarrow$(ii): We will show: If $\mathfrak{a}$ is an abelian ideal, then $\mathfrak{a} \subseteq \mathfrak{g}^\perp = \{x \in \mathfrak{g} : (x, \mathfrak{a}) = 0\}$ where $(\cdot,\cdot) = (\cdot,\cdot)_{\mathrm{ad}}$. Write $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$, $\mathfrak{h}$ a vector space complement to $\mathfrak{a}$. If $x \in \mathfrak{a}$, then $\mathrm{ad}(\mathfrak{a})$ has matrix

$$\begin{array}{c} \mathfrak{a} \\ \mathfrak{h} \end{array} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

as $\mathfrak{a}$ abelian and an ideal. If $x \in \mathfrak{g}$, then $\mathrm{ad}(x)$ has matrix

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

as $\mathfrak{a}$ is an ideal, so

$$\operatorname{tr}(\operatorname{ad}\mathfrak{a}, \operatorname{ad} x) = \operatorname{tr}\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = 0$$

so $(\mathfrak{a}, \mathfrak{g})_{\mathrm{ad}} = 0$.
(ii)$\Rightarrow$(iii): Let $\mathfrak{i} = \mathfrak{g}^\perp$, which is an ideal. Suppose $\mathfrak{i} \neq 0$, then $\mathrm{ad}\colon \mathfrak{i} \to \mathfrak{gl}(\mathfrak{g})$ has $(x, y)_{\mathrm{ad}} = 0$ for all $x, y \in \mathfrak{i}$. Now, by Cartan's criterion $\mathfrak{i}/Z(\mathfrak{i})$ is solvable, so $\mathfrak{i}$ is solvable.
(ii),(iii)$\Rightarrow$(i): Let $(\cdot,\cdot)_{\mathrm{ad}}$ be non-degenerate. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a minimal ideal.
Claim: $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is either 0 or non-degenerate.
Proof: $\{x \in \mathfrak{a} : (x, \mathfrak{a}) = 0\} = \mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal. But $\mathfrak{a}$ is minimal, so $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$ or $\mathfrak{a}$. □
But Cartan implies $\mathfrak{a}$ is solvable if $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is zero. But $R(\mathfrak{g}) = 0$, so it must be $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ non-degenerate. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, as $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ and $(\cdot,\cdot)_{\mathrm{ad}}$ are non-degenerate, with $\mathfrak{a}$ simple. As this is a direct sum of Lie algebras, any ideal of $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. Inductively repeating this with $\mathfrak{a}^\perp$ instead of $\mathfrak{g}$ gives $\mathfrak{g} = \bigoplus \mathfrak{a}_i$ where $\mathfrak{a}_i$ are simple Lie algebras (minimal and ideals).
(i)$\Rightarrow$(ii): Claim: If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is a direct sum of its minimal ideals in a *unique* manner. To prove this, note first that all the components of the direct sum are ideals in $\mathfrak{g}$. Write $\mathfrak{g} = \bigoplus \mathfrak{a}_i$. Assume that $\mathfrak{r} \subseteq \mathfrak{g}$ is an minimal ideal. Consider $\mathfrak{r} \cap \mathfrak{a}_\mathfrak{i}$. These are either 0 or $\mathfrak{a}_i$, since the $\mathfrak{a}_i$ are minimal. Hence, find j s.t. $\mathfrak{a}_\mathfrak{j} = \mathfrak{r}$. □
Now, by Cartan's criterion, we have $\mathfrak{a}$ is solvable if and only if $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{a}}$ is zero. But that would contradict the direct sum composition into minimal ideals (since then $\mathfrak{a} \subset \mathfrak{a}^\perp$). Hence, $R(\mathfrak{a}) = 0$.

Finally, let $D : \mathfrak{g} \to \mathfrak{g}$ be a derivation, $\mathfrak{g}$ semisimple. Consider the linear function $l : \mathfrak{g} \to K$ with $x \mapsto \operatorname{tr}_{\mathfrak{g}}(D(\operatorname{ad} x))$. As $\mathfrak{g}$ is semisimple, $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate, so there exists $y \in \mathfrak{g}$, st. $l(x) = (y, x)_{\mathrm{ad}}$ for all $x \in \mathfrak{g}$ (this follows from $x \mapsto \mathrm{ad}(x)$ being an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ (as a linear map with trivial kernel)). So we will show $E = D - \operatorname{ad} y$ is zero, i.e. $D = \operatorname{ad} y$. (Note E is a derivation). So to prove $Ea = 0$ for all $a \in \mathfrak{g}$, it is enough to show $(Ex, z)_{\mathrm{ad}} = 0$ for all x, z.

Assume $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate. Observe that $\mathrm{ad}(Ex) = E\cdot \mathrm{ad}\, x - \mathrm{ad}\, x\cdot E = [E, \mathrm{ad}\, x] : \mathfrak{g}\to\mathfrak{g}$ (as $\mathrm{ad}(Ex)(y) = [Ex, y] = E[x,y] - [x, Ey]$ as E is a derivation), so

$$\begin{aligned}(Ex, z)_{\mathrm{ad}} &= \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(Ex)\cdot \mathrm{ad}\, z) = \mathrm{tr}_{\mathfrak{g}}([E, \mathrm{ad}\, x]\cdot \mathrm{ad}\, z)\\ &= \mathrm{tr}_{\mathfrak{g}}(E[\mathrm{ad}\, x, \mathrm{ad}\, z]) = \mathrm{tr}_{\mathfrak{g}}(E\cdot \mathrm{ad}[x,z]) = 0,\end{aligned}$$

as by the definition of E: $\mathrm{tr}_{\mathfrak{g}}(E\cdot \mathrm{ad}(a)) = \mathrm{tr}_{\mathfrak{g}}(D\cdot \mathrm{ad}(a)) - l(a) = 0$. □

Exercise 3.28: Show that $[R(\mathfrak{g}), R(\mathfrak{g})] \subseteq \mathfrak{g}^{\perp} \subseteq R(\mathfrak{g})$.

Remark 3.29: If $\mathfrak{g}$ is any Lie algebra, then

$$0 \to \underbrace{R(\mathfrak{g})}_{\text{solvable ideal}} \to \mathfrak{g} \to \underbrace{\mathfrak{g}/R(\mathfrak{g})}_{\text{semisimple}} \to 0$$

is an exact sequence with maximal semisimple quotient.

Theorem 3.30 (Levi's theorem): If $\mathrm{char}\, k = 0$, this exact sequence splits, i.e. there exists a subalgebra $\mathfrak{s}\subseteq\mathfrak{g}$ isomorphic to $\mathfrak{g}/R(\mathfrak{g})$ (this algebra is not canonical), so we have $\mathfrak{g} = \mathfrak{s}\ltimes R(\mathfrak{g})$ (semidirect product). This is false in characteristic p.

Exercise 3.31:

(i) Let $\mathfrak{g} = \mathfrak{sl}_p\left(\overline{\mathbb{F}}_p\right)$. Show that $R(\mathfrak{g}) = \overline{\mathbb{F}}_p I$, but there is no complement.

(ii) A nilpotent Lie algebra always has non-inner derivations.

(iii) Let $\mathfrak{g} = \langle a, b\rangle$ with $[a,b] = b$. Show that $\mathfrak{g}$ has only inner derivations. Note that for this example $(\cdot,\cdot)_{\mathrm{ad}} = 0$, so this is an example showing that the condition that all derivations are inner does not imply that the Lie algebra is semisimple.

(iv) Let $\mathfrak{g}$ be a simple Lie Algebra above field a k, $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ two non-degenerate invariant bilinear forms. Show that there exists a $\lambda\in k^*$ st. $(\cdot,\cdot)_1 = \lambda(\cdot,\cdot)_2$

(v) Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ (assume this is simple). Define $(A,B) = \mathrm{tr}(AB)$, so $(A,B) = \lambda(A,B)_{\mathrm{ad}}$. Compute λ.

4 Structure Theory

In this section, we consider finite-dimensional Lie algebras.

Definition 4.1: A *torus* $\mathfrak{t} \subseteq \mathfrak{g}$ is an abelian subalgebra s.t. for all $t \in \mathfrak{t}$, $\operatorname{ad} t = [t, \cdot] \colon \mathfrak{g} \to \mathfrak{g}$ is a diagonalizable (i.e. semisimple) linear map. A *maximal torus* is a torus not contained in a bigger torus. A maximal torus is also called a *Cartan subalgebra*.

Example 4.2: Let $T = (S^1)^r \hookrightarrow G$ a compact Lie group (or $T = (\mathbb{C}^*)^r \hookrightarrow G$ a reductive algebraic group). Then $\mathfrak{t} = \operatorname{Lie} T \subseteq \operatorname{Lie} G$ is a torus, and maximal if T is.

Exercise 4.3:

(i) $\mathfrak{g} \subset \mathfrak{sl}_n$ or $\mathfrak{gl}_n$, $\mathfrak{t}$ be the set of diagonal matrices (or the matrices of trace 0 if in $\mathfrak{sl}_n$), then $\mathfrak{t}$ is a maximal torus.

(ii) $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right) \subseteq \mathfrak{sl}_2$ is *not* a torus.

Proof:

(i) Case $\mathfrak{t} \subseteq \mathfrak{gl}_n$ first: Clearly, $\mathfrak{t}$ is an abelian Lie subalgebra of $\mathfrak{gl}_n$. Moreover, if we choose the basis $\{E_{ij}, E_{ll} - E_{kk} \mid i \neq j, l < k\}$ of $\mathfrak{sl}_\mathfrak{n}$, we notice that for $D = \operatorname{Diag}(\lambda_1, \ldots, \lambda_n)$ we have $\operatorname{ad} D(E_{ij}) = (\lambda_i - \lambda_j)E_{ij}$ and therefore $\operatorname{ad} D$ is represented by a diagonal matrix. $\mathfrak{t}$ is maximal since if $\mathfrak{t}$ contains any other matrix (w.l.o.g. take $E_{ij}, i \neq j$) then $[t, E_{ij}] = (t_i - t_j)E_{ij} \neq 0$, for a suitable choice of t. Hence, $\mathfrak{t}$ is not abelian.

(ii) $\operatorname{ad}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ which is not diagonalisable as the eigenspace of the only eigenvector 0 is 2-dimensional. □

Let $t_1, \ldots, t_r \colon V \to V$ be pairwise commuting ($t_i t_j = t_j t_i$) diagonalizable linear maps. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$. Set $V_\lambda = \{v \in V \mid t_i v = \lambda_i v,\ \forall i = 1, \ldots, r\}$ simultaneous eigenspaces of all t_i.

Lemma 4.4: $V = \bigoplus_{\lambda \in (\mathbb{C}^r)^*} V_\lambda$, i.e. V breaks up into a direct sum of simultaneous eigenspaces.

Proof: Induction on r. If $r = 1$, this is clear by requiring that t_i is diagonalizable for all i.
If $r > 1$ consider $t_1, \ldots, t_{r-1}$, $V = \bigoplus V_{\lambda_1, \ldots, \lambda_{r-1}}$ by induction hypothesis. Now decompose $V_{\lambda_1, \ldots, \lambda_{r-1}}$ into eigenspaces for t_r (possible since t_r diagonalizable). □

Set $\mathfrak{t}$ to be the r-dimensional abelian Lie algebra with basis $t_1, \ldots, t_r$. Then V is a semisimple (that is completely reducible) representation of $\mathfrak{t}$, by Lemma 4.4, and $V = \bigoplus V_\lambda$ is its decomposition into isotypical (i.e. direct sums of isomorphic summands) representations.

Exercise 4.5: Show that every irreducible representation of $\mathfrak{t}$ is one dimensional.

Solution: If a subrepresentation W is not one-dimensional (as a vector space) then take $0 \neq v \in W$ and $\langle v \rangle$ is a $\mathfrak{t}$-subrepresentation of W. □

Set $\mathbb{C}_\lambda$ to be the one-dimensional representation of $\mathfrak{t}$, where $t_i \cdot w = \lambda_i w$, for all i. Then V_λ is a sum of $\dim V_\lambda$ copies of $\mathbb{C}_\lambda$ (a direct sum), and $\lambda \neq \mu$ implies $\mathbb{C}_\lambda \ncong \mathbb{C}_\mu$. Really, λ is a linear map $\mathfrak{t} \to \mathbb{C}$, i.e. $\lambda \in \mathfrak{t}$, where $\lambda(t_i) =: \lambda_i$. So, one-dimensional representations of $\mathfrak{t}$ correspond to irreducible representations of $\mathfrak{t}$ which are in 1-1 correspondence to elements of $\mathfrak{t}^* = \mathrm{Hom}_{\mathrm{Vect}}(\mathfrak{t}, \mathbb{C})$, and $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$, $V_\lambda = \{v \in V \mid t \cdot v = \lambda(t)v,\ \forall t \in \mathfrak{t}\}$ is called the *weight space decomposition* of V. Now, let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{t}$ a maximal torus. The weight space decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{g}_0 + \bigoplus_{\substack{\lambda \in \mathfrak{t}^* \\ \lambda \neq 0}} \mathfrak{g}_\lambda, \tag{6}$$

where $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [\mathfrak{t}, x] = 0\}$, $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x\ \forall t \in \mathfrak{t}\}$.

Definition 4.6: $R = \{\lambda \in \mathfrak{t}^* \mid \mathfrak{g}_\lambda \neq 0, \lambda \neq 0\}$ are the *roots* of $\mathfrak{g}$.

Example 4.7 (Essential): $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{t}$ the diagonal matrices in $\mathfrak{sl}_n$. If

$$t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}, \quad E_{ij} = (\delta_{i,k}\delta_{j,l})_{k,l},$$

then $[t, E_{ij}] = (t_i - t_j)E_{ij}$. Define $\varepsilon_i(t) := t_i$, so $\varepsilon_i \colon \mathfrak{t} \to \mathbb{C}$, i.e. $\varepsilon_i \in \mathfrak{t}^*$ and $\varepsilon_1, ..., \varepsilon_n$ span $\mathfrak{t}^*$, but $\varepsilon_1 + ... + \varepsilon_n = 0$ (as $\mathfrak{t} \subseteq \mathfrak{sl}_n$). So $[t, E_{ij}] = (\varepsilon_i - \varepsilon_j)(t)E_{ij}$ and so

$$R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}, \quad \mathfrak{g}_0 = \mathfrak{t}.$$

R are the roots of $\mathfrak{t}$. (This shows also that $\mathfrak{t}$ is a maximal torus), and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij},\ i \neq j$, is one-dimensional. So

$$\mathfrak{sl}_n = \mathfrak{t} \oplus \bigoplus_{\varepsilon_i - \varepsilon_j \in R} \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$$

is the root space decomposition of $\mathfrak{sl}_n$.

Exercise 4.8 (Exam!): Compute the root space decomposition for $\mathfrak{g} = \mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, where $\mathfrak{t} = \{\text{diagonal matrices}\} \cap \mathfrak{g}$, and

$$\mathfrak{so}_n = \{A \in \mathfrak{gl}_n \mid JA + A^T J = 0\}, \quad J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid MA + A^T M = 0\}, \quad M = \begin{pmatrix} 0 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & \iddots & & & \ddots & \\ -1 & & & & & 0 \end{pmatrix}.$$

In particular, show that $\mathfrak{t}$ is maximal torus and the root spaces are one-dimensional.

(i) Show $A \in \mathfrak{so}_n(\mathbb{C}) \iff A$ is skew-symmetric w.r.t. side diagonal.

(ii) Show $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right) \in \mathfrak{sp}_{2n}(\mathbb{C}) \iff A_2, A_3$ symmetric w.r.t. side diagonal, and A_4 is $-A_1$ transposed with side diagonal.

(iii) $\mathfrak{h} = \{\mathrm{Diag}(a_1, ..., a_n, 0, -a_n, ..., -a_1) \mid a_i \in K\}$ is a Cartan subalgebra in both cases.

(iv) $\mathfrak{so}_n$: $\mathbb{C}(E_{ij} - E_{n-j+1,n-i+1})$, $i+j < n+1$, $i \neq j$ are the root spaces for $\mathfrak{so}_n$. If $n = 2l$, roots are $R = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i,j \leq l, i \neq j\}$. If $n = 2l+1$, the roots are $R = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \leq i,j \leq l, i \neq j\}$.

(v) $\mathfrak{sp}_{2n}(\mathbb{C})$: root spaces are $\mathbb{C}(E_{ij} - E_{n-j+1,n-i+1})$, $1 \leq i,j \leq l$, $\mathbb{C}(E_{ij} + E_{n-j+1,n-i+1})$, $l < i \leq 2l, j \leq l$ or $i \leq l, l < j \leq 2l$ and roots are $R = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i,j \leq l, i \neq j\}$

(vi) Show $\mathfrak{sp}_{2l}$ $(l \geq 2)$ and $\mathfrak{so}_n$ are simple $(n > 4, n = 3)$.

(vii) Show $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$, $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, and $\mathfrak{so}_2 \cong \mathfrak{sp}_2 \cong \mathbb{C}$. Further, we have the isomorphisms $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, $\mathfrak{so}_6 \cong \mathfrak{sl}_4$.

Remark: all these root spaces are one-dimensional.

Solution:

(i) Consider the ij-th element:

$$\begin{aligned}(JA + A^T J)_{ij} &= \sum_{k=1}^{n} J_{ik} A_{kj} + A_{ki} J_{kj} \\ &= A_{n-i+1,j} + A_{n-j+1,i} = 0,\end{aligned}$$

i.e. A is skew symmetric w.r.t. the side diagonal.

(ii) $MA + A^T M = 0 \iff MAM^T = -A^T$. If $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right)$ this means that

$$\begin{pmatrix} JA_4J & -JA_3J \\ -JA_2J & JA_1J \end{pmatrix} = \begin{pmatrix} -A_1^T & -A_3T \\ -A_2^T & A_4^T \end{pmatrix},$$

looking at the four squares gives the claimed identities.

(iii) Note that $J\mathrm{Diag}(a_1, ..., a_l, 0, -a_l, ..., -a_1)^T J = \mathrm{Diag}(-a_1, ..., -a_l, 0, a_l, ..., a_1)$. Thus, $\mathfrak{t} \subseteq \mathfrak{so}_{2l+1}$. For $n = 2l$ consider the diagonal matrices of the form $\mathrm{Diag}(a_1, ..., a_l, -a_l, ..., -a_1)$. Also $M\mathrm{Diag}(a_1, ..., a_l, -a_l, ..., -a_1)^T M^T = \mathrm{Diag}(-a_1, ..., -a_l, a_l, ..., a_1)$ and hence $\mathfrak{t} \subseteq \mathfrak{sp}_{2l}$. Clearly, $\mathfrak{h}$ is abelian (diagonal matrices commute). Further, for any diagonal matrix $t = (t_1, ..., t_n)$, $[t, E_{ij}] = (t_i - t_j)E_{ij}$. Hence, $\mathrm{ad}\, t$ is diagonal for all $h \in \mathfrak{h}$. It remains to show that $\mathfrak{h}$ is maximal; this follows from the fact that the diagonal matrices form a maximal torus in $\mathfrak{gl}_n$.

(iv) Consider $\mathfrak{so}_n$. We have a basis $\{E_{i,j} - E_{n-j+1,n-i+1} : i + j \leq n\}$ for $\mathfrak{so}_n$. First, consider

$\mathfrak{so}_{2l}$, then

$$\begin{aligned}
[t, E_{i,j} - E_{n-j+1,n-i+1}] &= [t, E_{i,j}] - [t, E_{n-j+1,n-i+1}] \\
&= (t_i - t_j)E_{i,j} - (t_{n-j+1} - t_{n-i+1})E_{n-j+1,n-i+1} \\
&= \begin{cases} (a_i - a_j)E_{i,j} - (-a_j + a_i)E_{n-j+1,n-i+1} & \text{if } i, j \leq l \\ (-a_i - a_j)E_{i,j} - (-a_j - a_i)E_{n-j+1,n-i+1} & \text{if } j \leq l, i > l \\ (a_i + a_j)E_{i,j} - (a_j + a_i)E_{n-j+1,n-i+1} & \text{if } i \leq l, j > l \end{cases} \\
&= \begin{cases} (a_i - a_j)(E_{i,j} - E_{n-j+1,n-i+1}) & \text{if } i, j \leq l \\ (-a_i - a_j)(E_{i,j} - E_{n-j+1,n-i+1}) & \text{if } j \leq l, i > l\ . \\ (a_i + a_j)(E_{i,j} - E_{n-j+1,n-i+1}) & \text{if } i \leq l, j > l \end{cases}
\end{aligned}$$

This gives root spaces of the claimed form, with roots $\{\pm\varepsilon_i \pm \varepsilon_j : i + j < n, i < j\}$. If we consider $\mathfrak{so}_{2l+1}$, we have the additional equations

$$[t, E_{i,l+1} - E_{n-l,n-i+1}] = \pm(a_i \pm 0)(E_{i,l+1} - E_{n-l,n-i+1}),$$

giving the roots $\{\varepsilon_i, i = 1, ..., l\}$.

(v) For $\mathfrak{sp}_{2l}$ we have the following basis:

$$\begin{aligned}
E_{ij} - E_{n-j+1,n-i+1}, &\quad 1 \leq i, j \leq l, \\
E_{ij} + E_{n-j+1,n-i+1}, &\quad i \leq l, j > l \text{ or } i > l, j \leq l, \text{ and } i + j \leq 2l, \\
E_{i,n-i+1}, &\quad i = 1, \ldots, n.
\end{aligned}$$

Then a similar calculation as in (iv) shows that

$$[t, E_{i,j} - E_{n-j+1,n-i+1}] = (a_i - a_j)(E_{i,j} - E_{n-j+1,n-i+1}),$$

giving the root spaces $\mathbb{C}(E_{i,j} - E_{n-j+1,n-i+1})$, (w.r.t. root $\pm\varepsilon_i \pm \varepsilon_j$), $1 \leq i, j \leq l, i \neq j$. Further,

$$\begin{aligned}
[t, E_{ij} + E_{n-j+1,n-i+1}] &= (t_i + t_j)E_{ij} + (t_{n-j+1} + t_{n-i+1})E_{n-j+1,n-i+1} \\
&= (t_i + t_j)(E_{ij} + E_{n-j+1,n-i+1}),
\end{aligned}$$

this gives the root spaces $\mathbb{C}(E_{i,j} + E_{n-j+1,n-i+1})$ (w.r.t. root $\varepsilon_i + \varepsilon_j$), for $i \leq l, j > l, i + j \leq 2l$, and for $i > l, j \leq l, i + j \leq 2l$ we obtain the roots $-(\varepsilon_i + \varepsilon_j)$. Finally, $[t, E_{i,n-i+1}] = 2t_i E_{i,n-i+1} = \varepsilon_i(t)E_{i,n-i+1}$. □

Proposition 4.9: The Lie algebra $\mathfrak{sl}_n$ is simple, for $n \geq 2$.

Proof: Recall $\mathfrak{sl}_n = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $R = \{\varepsilon_i - \varepsilon_j : i \neq j\}$, $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$. Suppose $\mathfrak{a} \subseteq \mathfrak{sl}_n$ is a non-zero ideal. Choose $r \in \mathfrak{a}, r \neq 0$, s.t. if we write $r = t + \sum_{\alpha \in R} e_\alpha$ with $e_\alpha \in \mathfrak{g}_\alpha$, then the number of non-zero terms is minimal.

Now suppose $t \neq 0$. Choose $t_0 \in \mathfrak{t}$, st. $\alpha(t_0) \neq 0$ for all $\alpha \in R$ (i.e. t_0 has distinct eigenvalues). Consider $[t_0, r] = \sum_{\alpha \in R} \alpha(t_0) \cdot e_\alpha \in \mathfrak{a}$, as $\mathfrak{a}$ is an ideal. This, if non-zero, has fewer terms than r, contradicting our choice of r, hence must be zero. Therefore $e_\alpha = 0$ for all $\alpha \in R$ and so $r = t \in \mathfrak{t}$, $t \neq 0$. Now this implies that there exists an $\alpha \in R$ with $\alpha = \varepsilon_i - \varepsilon_j$ s.t. $\alpha(t) =: c \neq 0$. Hence $cE_{ij} = [t, E_{ij}] \in \mathfrak{a}$, as $\mathfrak{a}$ is an ideal with $c \neq 0$, so $E_{ij} \in \mathfrak{a}$. But now

$[E_{ij}, E_{jk}] = E_{ik}$ if $i \neq k$ and $[E_{si}, E_{ij}] = E_{sj}$ if $j \neq s$, so $E_{ij} \in \mathfrak{a}$ implies $E_{ab} \in \mathfrak{a}$, for all $a \neq b$. But now $E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}] \in \mathfrak{a}$ also, but $\{E_{ab},\ E_{ii} - E_{i+1,i+1}\}$ forms a basis for $\mathfrak{sl}_n$, so $\mathfrak{a} = \mathfrak{sl}_n$.

If $t = 0$, write $r = \sum_{\alpha \in R} e_\alpha$, and if there is only one non-zero term, then $r = cE_{ij}$, $c \neq 0$, argue as before, to get $\mathfrak{a} = \mathfrak{sl}_n$. So $r = cE_\alpha + dE_\beta + \sum_{\gamma \in R \setminus \{\alpha,\beta\}} e_\gamma$ with α, β distinct. Choose $t_0 \in \mathfrak{t}$ s.t. $\alpha(t_0) \neq \beta(t_0)$. Then a suitable linear combination of $[t_0, r]$ and r has fewer terms than r, contradicting our choice. □

Proposition 4.10: Let $\mathfrak{g}$ be a semisimple Lie algebra. Then maximal tori exist, i.e if $\mathfrak{t}$ is a maximal torus, then $\mathfrak{t} \neq 0$. Moreover $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [\mathfrak{t}, x] = 0\} = \mathfrak{t}$.

Proof: omitted □

This means that the root space decomposition of a semisimple Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Theorem 4.11 (Structure theorem for semisimple Lie algebras): Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ maximal torus, write $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then:

(i) $\mathbb{C}R = \mathfrak{t}^*$, i.e. the roots span $\mathfrak{t}^*$,

(ii) $\dim \mathfrak{g}_\alpha = 1$,

(iii) If $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. If $\alpha + \beta \notin R$, and $\alpha \neq -\beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(iv) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ and is one-dimensional, and $\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ is a Lie subalgebra, isomorphic to $\mathfrak{sl}_2$.

Proof:

(i) If not, there is some $t \in \mathfrak{t}$, $t \neq 0$ with $\alpha(t) = 0$ for all $\alpha \in R$. But then for $x \in \mathfrak{g}_\alpha$, we have $[t, x] = 0$, i.e. $[t, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$. But $[t, \mathfrak{t}] = 0$, as $\mathfrak{t}$ is abelian. So t is in the center of $\mathfrak{g}$. But $\mathfrak{g}$ is semisimple, so it has no abelian ideals and therefore no center.

Next we will prove several properties, which lead to the proof of the theorem, but will not be directly assigned to its statements.

(a) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{t}^*$

Proof. By the Jacobi identity for all $t \in \mathfrak{t}, x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$ we have

$$[t, [x, y]] = [[t, x], y] + [x, [t, y]] = \lambda(t)[x, y] + \mu(t)[x, y] = (\lambda + \mu)(t)[x, y]$$ □

(b) $(\mathfrak{g}_\lambda, \mathfrak{g}_\mu)_{\mathrm{ad}} = 0$ if $\lambda \neq -\mu$. Moreover, $(\cdot,\cdot)_{\mathrm{ad}}|_{\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}}$ is non-degenerate.

Proof. Let $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$. By (a) $(\mathrm{ad}(x)\,\mathrm{ad}(y))^N \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\alpha+N(\lambda+\mu)}$, but $\lambda + \mu \neq 0$ and $\mathfrak{g}$ is finite-dimensional. So for N sufficiently large, we have $\alpha + N(\lambda + \mu) \notin R$ and then $\mathfrak{g}_{\alpha+N(\lambda+\mu)} = 0$. So $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is nilpotent and therefore

$$\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = (x, y)_{\mathrm{ad}} = 0.$$

On the other hand, $(\cdot,\cdot)_{\text{ad}}$ is non-degenerate (by (3.27) as $\mathfrak{g}$ is semisimple) and $\mathfrak{g} = \bigoplus_{\lambda\in\mathfrak{t}^*}\mathfrak{g}_\lambda$, so it must be that $(\cdot,\cdot)_{\text{ad}}|_{\mathfrak{g}_\lambda+\mathfrak{g}_\mu}$ is non-degenerate, so in particular $(\cdot,\cdot)_{\text{ad}}|_{\mathfrak{g}_\lambda+\mathfrak{g}_{-\lambda}}$ is non-degenerate. ☐

(c) In particular, $(\cdot,\cdot)_{\text{ad}}|_{\mathfrak{t}}$ is non-degenerate ($\mathfrak{t}=\mathfrak{g}_0$) (warning: this is not $(\cdot,\cdot)_{\text{ad}\,\mathfrak{t}}$, which is 0), so it defines an isomorphism $\nu\colon \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$, $\nu(t)(t') = (t,t')_{\text{ad}}$ and equippes $\mathfrak{t}^*$ with the induced inner product, i.e. $(\nu(t),\nu(t'))_{\text{ad}} := (t,t')_{\text{ad}}$.

(d) If $\alpha\in R$, then $-\alpha\in R$

Proof. $(\cdot,\cdot)_{\text{ad}}$ is non-degenerate on $\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}$. But by (b) $(\mathfrak{g}_\alpha,\mathfrak{g}_\alpha)_{\text{ad}} = 0$ if $\alpha\neq 0$. This implies $\mathfrak{g}_{-\alpha}$ is non-zero (and isomorphic to $(\mathfrak{g}_\alpha)^*$). ☐

(e) If $x\in\mathfrak{g}_\alpha$, $y\in\mathfrak{g}_{-\alpha}$, then

$$[x,y] = (x,y)_{\text{ad}}\cdot\nu^{-1}(\alpha)\in\mathfrak{t}. \tag{7}$$

Proof. $\nu([x,y])\in\mathfrak{t}^*$, so it is determined by $([x,y],t)_{\text{ad}}$ (7) follows from

$$\nu([x,y])(t) = ([x,y],t)_{\text{ad}} \overset{(*)}{=} (t,[x,y])_{\text{ad}} = (x,y)_{\text{ad}}\cdot\alpha(t),$$

$(*)$ as $(\cdot,\cdot)_{\text{ad}}$ is an invariant form. ☐

(f) Let $e_\alpha\in\mathfrak{g}_\alpha$ be non-zero, and pick $e_{-\alpha}\in\mathfrak{g}_{-\alpha}$ s.t. $(e_\alpha,e_{-\alpha})_{\text{ad}}\neq 0$ (possible as $(\cdot,\cdot)_{\text{ad}}|_{\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}}$ is non-degenerate), so $[e_\alpha,e_{-\alpha}] = (e_\alpha,e_{-\alpha})_{\text{ad}}\nu^{-1}(\alpha)$, by (7). This implies

$$[\nu^{-1}(\alpha),e_{\pm\alpha}] = \pm\alpha(\nu^{-1}(\alpha))e_{\pm\alpha} = \pm(\nu^{-1}(\alpha),\nu^{-1}(\alpha))e_\alpha = \pm(\alpha,\alpha)e_\alpha.$$

Claim: $(\alpha,\alpha)\neq 0$.

Proof. Suppose $(\alpha,\alpha)=0$. Put $\mathfrak{m} := \langle e_\alpha,e_{-\alpha},\nu^{-1}(\alpha)\rangle$. Then $[\mathfrak{m},\mathfrak{m}] = \mathbb{C}\nu^{-1}(\alpha)$, and so $\mathfrak{m}$ is solvable. But then Lie's Theorem implies that $\operatorname{ad}[\mathfrak{m},\mathfrak{m}]$ acts by nilpotent operators on $\mathfrak{g}$, i.e. $\operatorname{ad}\nu^{-1}(\alpha)$ is nilpotent ($\Longleftrightarrow$ all eigenvalues are 0). But $\nu^{-1}(\alpha)\in\mathfrak{t}$, so acts diagonalizable, by definition. Hence $\nu^{-1}(\alpha)=0$, i.e. $\alpha=0$, contradiction. ☐

Therefore, we can define $h_\alpha = \frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)}\in\mathfrak{t}$ and rescale e_α so that $(e_\alpha,e_{-\alpha})_{\text{ad}} = \frac{2}{(\alpha,\alpha)}$. Exercise: Check that the linear map $\mathfrak{m}\to\mathfrak{sl}_2$ defined by $e_\alpha\mapsto e, e_{-\alpha}\mapsto f, h_\alpha\mapsto h$ is an isomorphism of Lie algebras.

(g) $\dim\mathfrak{g}_{-\alpha} = 1, \forall\alpha\in R$.

Proof. Pick $\mathfrak{m}_\alpha = \langle e_\alpha,h_\alpha,e_{-\alpha}\rangle$ as above, so $\mathfrak{m}_\alpha\cong\mathfrak{sl}_2$, and suppose $\dim\mathfrak{g}_{-\alpha}>1$, then

$$\mathfrak{g}_{-\alpha}\to\mathbb{C}\nu^{-1}(\alpha),\ x\mapsto[e_\alpha,x]$$

must have a non-trivial kernel, i.e. there exists $v\in\mathfrak{g}_{-\alpha}$ s.t. $\operatorname{ad}(e_\alpha)v=0$, i.e. v is a highest weight vector with weight -2 as $\operatorname{ad}(h_\alpha)v = -\alpha(h_\alpha)v = -2v$ (by definition $\alpha(h_\alpha)=2$), but $\dim\mathfrak{g}<\infty$, so contradiction (highest weights of finite-dimensional $\mathfrak{sl}_2$ representations are in $\mathbb{N}$). ☐

Note that the proof of part (iii) is still incomplete. It will be proven in the following theorem, which continues the structure theorem. □

Exercise 4.12: Check Theorem 4.11 for the classical Lie algebras $\mathfrak{sl}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$.

Theorem 4.13 (Structure Theorem continued):

(v) If $\alpha, \beta \in R$, then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(vi) If $\alpha \in R$ and $k\alpha \in R$, then $k = \pm 1$.

(vii) $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is an irreducible module for $(\mathfrak{sl}_2)_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha}\rangle$. In particular, the set

$$\{k\alpha + \beta \mid k\alpha + \beta \in R, k \in \mathbb{Z}\} \cup \{0\}$$

is of the form $\beta - p\alpha, \beta - (p-1)\alpha, ..., \beta + q\alpha$, where $p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, the *$\alpha$-string through β*.

Proof:

(v) Let $q = \max\{k \mid \beta + k\alpha \in R\}$, $v \in \mathfrak{g}_{\beta+q\alpha}$, $v \neq 0$. Then $[e_\alpha, v] \in \mathfrak{g}_{\beta+(q+1)\alpha} = 0$, and $[h_\alpha, v] = (\beta + q\alpha)(h_\alpha)v$. But

$$(\beta + q\alpha)\underbrace{\frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)}}_{h_\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} + 2q =: N,$$

so v is a highest weight vector with weight $N \in \mathbb{N}$, as $q \in \mathbb{N}$, this implies $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(vii) Structure of $\mathfrak{sl}_2$-modules tells us that $(\operatorname{ad} e_{-\alpha})^r \neq 0$, if $0 \leq r \leq \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2q = N$, and $(\operatorname{ad} e_{-\alpha})^{N+1} v = 0$. It follows that $\{\beta + q\alpha - k\alpha : 0 \leq k \leq N\} \cup \{0\}$ are all roots. So $\beta + q\alpha, \beta + (q-1)\alpha, ..., \beta - \left(q + \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right)\alpha$ are roots. So, we show that there are no more roots.

Let $p = \max\{k \mid \beta - k\alpha \in R\}$, and $w \in \mathfrak{g}_{\beta-p\alpha}$, $w \neq 0$, then $[e_{-\alpha}, w] = 0$, $[h_\alpha, w] = \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} - 2p\right) \cdot w$ is the lowest weight vector of an $(\mathfrak{sl}_2)_\alpha$-module, so we get that

$$\beta - p\alpha, \beta - (p-1)\alpha, ..., \beta + \left(p - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right)\alpha$$

are all roots. Put $p' = q + \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, so $p' \leq p$ by definition of p, i.e. $q + \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \leq p$, and by definition of q, $p - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \leq q$. Hence equality.

(vi) If $k\alpha$ is a root, then as is (v) $\frac{2(\alpha,k\alpha)}{(k\alpha,k\alpha)} = \frac{2}{k} \in \mathbb{Z}$, and $\frac{2(\alpha,k\alpha)}{(\alpha,\alpha)} = 2k \in \mathbb{Z}$, so it is enough to show that $\alpha \in R$ implies $2\alpha \notin R$. If not, let $v \in \mathfrak{g}_{-2\alpha}$, $v \neq 0$. Then $\operatorname{ad} e_\alpha v \in \mathfrak{g}_{-\alpha}$, but this implies $\operatorname{ad}(e_\alpha)v = 0$, as $(\cdot,\cdot)_{\mathrm{ad}}$ is non-degenerate on $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathbb{C}e_\alpha + \mathbb{C}e_{-\alpha}$, so $v \in \mathfrak{g}_{-2\alpha}$ is a highest weight vector of weight -4, a contradiction.

(iii) Finally, we prove that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in R$. We have just shown that $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is irreducible $\mathfrak{m}_\alpha$-module, i.e. $\operatorname{ad}(e_\alpha)\colon \mathfrak{g}_{\beta+k\alpha} \to \mathfrak{g}_{\beta+(k+1)\alpha}$ is an isomorphism if $k < q$. But $\mathfrak{g}_{\alpha+\beta} \neq 0$ implies $q \geq 1$, so $\operatorname{ad}(e_\alpha)\mathfrak{g}_\beta = \mathfrak{g}_{\beta+\alpha}$ □

Definition 4.14: For $\alpha \in \mathfrak{t}^*$, define the *reflection at* α as

$$s_\alpha \colon \mathfrak{t}^* \to \mathfrak{t}^*,\ s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

Claim 4.15: Property (vii) of Theorem 4.13 says $s_\alpha(\beta) \in R$, for all $\alpha, \beta \in R$.

Proof: Put $r = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. If $r \geq 0$, $p = q + r \geq r$. If $r \leq 0$, $q = p - r \geq -r$. In both cases, we get $\beta - r\alpha$ in the α-string through β. □

Proposition 4.16: Recall that R spans $\mathfrak{t}^*$.

(i) If $\alpha, \beta \in R$ then $(\alpha, \beta) \in \mathbb{Q}$.

(ii) If we pick a basis $\beta_1, \ldots, \beta_l$ of $\mathfrak{t}^*$ with $\beta_i \in R$ and $\beta \in R$, then $\beta = \sum c_i\beta_i$ with $c_i \in \mathbb{Q}$, i.e. $\dim_\mathbb{Q} R = \dim_\mathbb{C} \mathfrak{t}$.

(iii) $(\cdot,\cdot)$ is positive definite on $\mathbb{Q}R$.

Proof:

(i) Since $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, it is enough to show $(\beta,\beta) \in \mathbb{Q}$ if $\beta \in R$. Now let $h, h' \in \mathfrak{t}$. Then by structure theorem

$$(h,h')_{\mathrm{ad}} = \mathrm{tr}_\mathfrak{g}(\mathrm{ad}\, h\, \mathrm{ad}\, h') = \sum_{\alpha\in R} \alpha(h)\alpha(h')$$

So if $\lambda, \mu \in \mathfrak{t}^*$, we have

$$(\lambda,\mu) = (\nu^{-1}(\lambda), \nu^{-1}(\mu))_{\mathrm{ad}} = \sum_{\alpha\in R} \alpha(\nu^{-1}(\lambda))\alpha(\nu^{-1}(\mu)) = \sum_{\alpha\in R} (\lambda,\alpha)(\mu,\alpha)$$

so $(\beta,\beta) = \sum_{\alpha\in R}(\alpha,\beta)^2$. Dividing by $\frac{1}{4}(\beta,\beta)^2$ we get

$$\frac{4}{(\beta,\beta)} = \sum_{\alpha\in R} \left(\frac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2 \in \mathbb{Z} \Rightarrow (\beta,\beta) \in \mathbb{Q}.$$

(ii) Let $\beta_1, \ldots, \beta_l$ be a basis of $\mathfrak{t}^*$ consisting of roots and let $B = ((\beta_i, \beta_j)_{ij}$ be the matrix of the bilinear form. Since $(\cdot,\cdot)$ is non-degenerate, $\det B \neq 0$. Now if $\beta = \sum c_i\beta_i \in R$, we have $(\beta, \beta_i) = \sum_j c_j(\beta_j, \beta_i)$ but

$$\begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} (\beta,\beta_1) \\ \vdots \\ (\beta,\beta_l) \end{pmatrix} \in \mathbb{Q}^l$$

(iii) If $\lambda \in \mathbb{Q}R$, then $\lambda = \sum c_i\beta_i$ with $c_i \in \mathbb{Q}$ by (ii), so $(\lambda, \alpha) \in \mathbb{Q}$ for all $\alpha \in R$, by (i). But then

$$(\lambda,\lambda) = \sum_{\alpha\in R} (\lambda,\alpha)^2 \geq 0.$$

And if $(\lambda,\lambda) = 0$, then $(\lambda,\alpha) = 0$ for all $\alpha \in R$, hence $\lambda = 0$ as R spans $\mathfrak{t}^*$ and $(\cdot,\cdot)$ is non-degenerate. □

Exercise 4.17: Let $(\cdot,\cdot)$ be a non-degenerate, bilinear, symmetric form and let B defined as in 4.16. Show that $\det B \neq 0$.

5 Root Systems

Definition 5.1: Let V be a vector space over $\mathbb{R}$. Let $(\cdot,\cdot)$ be an inner product (here it is a *positive definite*, bilinear, symmetric form). If $\alpha \in V$, $\alpha \neq 0$, write $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. Note that $(\alpha, \alpha^\vee) = 2$.
Define $s_\alpha : V \to V$ by $s_\alpha(v) = v - (v, \alpha^\vee)\alpha$ (which is a linear map).

Lemma 5.2: The linear map s_α is the reflection in the hyperplane orthogonal to α. In particular, all of its eigenvectors are 1, except of one which is -1. So $s_\alpha^2 = 1$ ($\iff$ $(s_\alpha+1)(s_\alpha-1) = 0$), and $s_\alpha = O(V, (\cdot,\cdot))$ the orthogonal group defined by $(\cdot,\cdot)$.

Proof: $V = \mathbb{R}\alpha \oplus \alpha^\perp$ where $\alpha^\perp = \{v \in V : (\alpha, v) = 0\}$. Furthermore, we have

$$s_\alpha(\alpha) = \alpha - (\alpha, \alpha^\vee) \cdot \alpha = \alpha - 2\alpha = -\alpha.$$

And for each $v \in \alpha^\perp$ we have

$$s_\alpha(v) = v - (v, \alpha^\vee) \cdot \alpha = v - \frac{2 \cdot \alpha}{(\alpha,\alpha)} \cdot \underbrace{(v, \alpha)}_{=0} = v. \qquad \square$$

Definition 5.3: A *root system* R in V is a finite set $R \subseteq V$ s.t.

(i) $0 \notin R$ and $\mathbb{R}R = V$ (i.e. $V = \mathrm{span}(R)$),

(ii) for all $\alpha, \beta \in R$ we have $(\alpha, \beta^\vee) \in \mathbb{Z}$,

(iii) $s_\alpha R \subseteq R$ for all $\alpha \in R$.

A root system is called *reduced* if

(iv) $\alpha, k\alpha \in R \implies k = \pm 1$.

Note that (iii) implies that $s_\alpha(\alpha) = -\alpha \in R$.

Example 5.4: If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ is its weight space decomposition, then $(R, \mathbb{R}R)$ is a reduced root system.

Definition 5.5: Let $W := \langle\{s_\alpha : \alpha \in R\}\rangle \subseteq GL(V)$. The group W is called the *Weyl group* of R.

Lemma 5.6: The Weyl group W is finite.

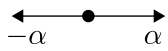
Proof: Since the s_α are invertible, and $s_\alpha R \subseteq R$ by 5.3(iii), each s_α permutes the elements of R which is finite. So there exists an embedding $W \hookrightarrow \mathrm{Sym}(|R|)$. But by 5.3(i) this map is an injection since $\mathbb{R}R = V$ and therefore, if s_α, s_β act equally on R, they coincide on the whole of V. $\square$

Definition 5.7: The *rank* of a root system $R \subset V$ is defined as $\dim_\mathbb{R} V$ as a vector space. An *isomorphism of root systems* $(R, V) \to (R', V')$ is a bijective linear map $\varphi\colon V \to V'$ s.t. $\varphi(R) = R'$. Note that φ is not required to be an isometry (i.e. does not have to preserve the inner product).

If (R, V) and (R', V') are root systems, so is $(R \coprod R', V \oplus V')$. A root system which is not isomorphic to a direct sum like this is called *irreducible*.

Example 5.8:

rk 1: A_1 : The only rank 1 root system is $V = \mathbb{R}$ with inner product $(x, y) = xy$ and roots $R = \{\alpha, -\alpha\}, \alpha \neq 0$. Its Weyl group is given by $W = \mathbb{Z}/2$. We call this root system A_1. This is the root system of $\mathfrak{sl}_2$.



rk 2: $A_1 \times A_1$: Take $V = \mathbb{R}^2$ with the usual inner product. Then $R = \{e_1, -e_1, e_2, -e_2\}$ with the standard basis vectors is a root system. Note that this is $A_1 \times A_1$ and therefore not irreducible. Here $W = \mathbb{Z}/2 \times \mathbb{Z}/2$.
A_2 : Let $\alpha = \alpha^\vee$, $\beta = \beta^\vee$, $(\alpha, \beta) = -1$. Then $W = S_3$. We call this root system A_2, it appears as the root system of $\mathfrak{sl}_3$.
B_2 : Let $\alpha = e_1$, $(\alpha, \alpha) = 1$, $\beta = e_2 - e_1$, $(\beta, \beta) = 2$, $\alpha, \alpha + \beta$ are short roots, $\beta, 2\alpha + \beta$ long roots. Then W is the symmetry group of the square, i.e. $W = D_8$, the dihedral group of order 8. This is the root system of $\mathfrak{sp}_4$ and $\mathfrak{so}_5$.
G_2 : Also D_{12} appears as Weyl group of a root system, called G_2.

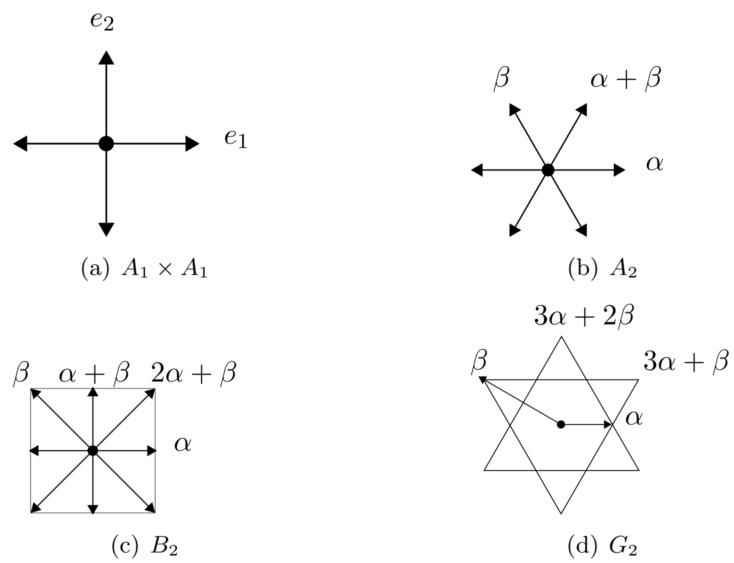


Figure 1: rank 2 root systems

Exercise 5.9: Check that all the above examples are root systems and that A_2, B_2, G_2 are the only irreducible roots systems of rank 2.

Lemma 5.10: Let R be a root system. Then $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is an root system.

Proof: Clearly, $0 \notin R^\vee$. Also, since $\alpha^\vee$ has the same direction as α, we get that $\mathbb{R}R = V$.

Notice $(\alpha^\vee)^\vee = \alpha$. Thus, for $\alpha, \beta \in R$, $(\alpha^\vee, (\beta^\vee)^\vee) = (\beta, \alpha^\vee) \in \mathbb{Z}$. Lastly,

$$\begin{aligned} s_{\alpha^\vee}(v) &= v - (v,\alpha)\alpha^\vee \\ &= v - (v,\alpha)\frac{2\alpha}{(\alpha,\alpha)} \\ &= v - (v, \frac{2\alpha}{(\alpha,\alpha)})\alpha \\ &= s_\alpha(v). \end{aligned}$$

Hence, $s_{\alpha^\vee}R = s_\alpha R \subset R$. □

Definition 5.11: R is *simply laced* if all the roots are of the same length (e.g. $A_1, A_1 \times A_1, A_2$, not B_2, G_2).

Exercise 5.12: If R is simply laced, then (R, V) is isomorphic to a root system (R', V') with $(\alpha, \alpha) = 2$, for all $\alpha \in R'$ (i.e. $\alpha = \alpha^\vee$).

Solution: Say $|(\alpha,\alpha)| = \sqrt{(\alpha,\alpha)} = \lambda$, for all $\alpha \in R$ as R is simply laced. Now define $\overline{\alpha} := \frac{\sqrt{2}}{\lambda}\alpha$ (if this is an root system, then it is isomorphic to R via multiplication by a scalar). Then $(\overline{\alpha}, \overline{\alpha}) = 2$, and $\overline{\alpha}^\vee = \overline{\alpha}$. $\overline{R} := \{\overline{\alpha} \mid \alpha \in R\}$ is also a root system. Clearly, $0 \notin \overline{R}, \mathbb{R}\overline{R} = V$. Further,

$$(\overline{\alpha}, \overline{\beta}^\vee) = \left(\frac{\sqrt{2}}{\lambda}\right)^2 \frac{\lambda^2}{2}(\alpha, \beta^\vee) = (\alpha, \beta^\vee) \in \mathbb{Z},$$

where we apply that R is simply laced. Lastly, notice that $s_{\overline{\alpha}} = s_\alpha$. □

Definition 5.13: A *lattice* L is a finitely generated free abelian group ($\cong \mathbb{Z}^l$) with bilinear form $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$ s.t $(L \otimes_{\mathbb{Z}} \mathbb{R}, (\cdot,\cdot))$ is an inner product space. A *root* of L is an $\alpha \in L$ with $(\alpha, \alpha) = 2$. Write

$$R_L = \{l \in L \mid (l,l) = 2\} = \{l \in L \mid l^\vee = l\}$$

for the set of roots of L. Note that $\alpha \in R_L$ implies $s_\alpha(L) \subseteq L$.

Lemma 5.14: The set of roots R_L is a root system in $\mathbb{R}R_L$. Moreover, it is simply laced.

Proof: Everything is obvious, except: R_L is finite. But R_L is the intersection of a compact set, the sphere $\{v \in \mathbb{R}L \mid (v,v) = 2\}$, with the discrete set L, so it is finite. □

We say L is *generated by roots* if $\mathbb{Z}R_L = L$. Note, this implies that L is an *even lattice*, i.e. $(l,l) \in 2\mathbb{Z}$ for all $l \in L$.

Example 5.15:

(i) Let $L = \mathbb{Z}\alpha$, $(\alpha,\alpha) = \lambda$. If $\lambda = 2$ and $R_L = \{\pm\alpha\}$, then L is generated by roots. If $\frac{k^2\lambda}{2} \neq 1$, for all $k \in \mathbb{Z}$, then $R_L = \emptyset$.

(ii) A_n: Consider $\mathbb{Z}^{n+1} = \bigoplus_{i=1}^{n+1} \mathbb{Z}e_1$ and $(e_i, e_j) = \delta_{ij}$ as a square lattice. Define

$$L = \{l \in \mathbb{Z}^{n+1} : (l, e_1 + \ldots + e_{n+1}) = 0\} = \left\{\sum_{i=0}^{n+1} a_i e_i : a_i \in \mathbb{Z}, \sum a_i = 0\right\} \cong \mathbb{Z}^n.$$

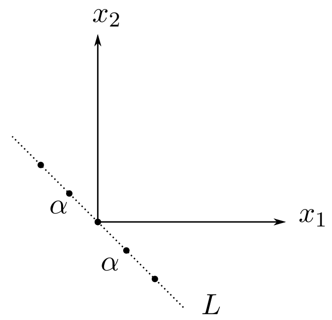
Now, $R_L = \{e_i - e_j : i \neq j\}$ and so $|R_L| = n(n+1)$, $\mathbb{Z}R_L = L$. If $\alpha = e_i - e_j$ then

$$\begin{aligned} s_\alpha\left(\sum_{k=1}^{n+1} x_k e_k\right) &= \sum_{k=1}^{n+1} x_k e_k - \left(\left(e_i, \sum_{k=1}^{n+1} x_k e_k\right) - \left(e_j, \sum_{k=1}^{n+1} x_k e_k\right)\right)(e_i - e_j) \\ &= \sum_{k=1}^{n+1} x_k e_k - (x_i - x_j)(e_i - e_j) \\ &= x_1 e_1 + ... + x_j e_i + ... + x_i e_j + ... + x_{n+1} e_{n+1}, \end{aligned}$$

i.e. $s_{e_i - e_j}$ swaps i^{th} and j^{th} coordinate. Hence

$$W = \langle s_{e_i - e_j} : i, j = 1, \ldots, n\rangle = S_{n+1},$$

the symmetric group of $n+1$ letters. Call $(R_L, \mathbb{R}L)$ root system of the type A_n, where n is the rank of the root system. Note that A_n is irreducible.
Exercise: Check these statements, then draw $L \subseteq \mathbb{Z}^{n+1}$ and R_L for $n = 1, 2$, check that these agree with A_1, A_2 as defined before. E.g. the roots system A_1 is:



Moreover, show that the root system of $\mathfrak{sl}_{n+1}$ is of type A_n.

(iii) D_n: Consider the square lattice $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_1$, with $(e_i, e_j) = \delta_{ij}$. Then $R_{\mathbb{Z}^n} = \{\pm e_i \pm e_j \mid i \neq j\}$. Set

$$L = \mathbb{Z}R_{\mathbb{Z}^n} = \left\{ l = \sum_{i=1}^n a_i e_i \;\middle|\; a_i \in \mathbb{Z}, \sum a_i \in 2\mathbb{Z} \text{ (i.e. even)} \right\},$$

then $s_{e_i - e_j}$ swaps the i-th and j-th component as before, and

$$s_{e_i + e_j}(x_1 e_1 + ... + x_n e_n) = x_1 x_1 + ... - x_j e_i ... - x_i e_j ... + x_n e_n,$$

i.e. $s_{e_i + e_j}$ swaps the i-th and j-th component and changes signs of these components. If L has this form, we say $(R_L, \mathbb{Z}R_L)$ is of type D_n. In this case, $|D_n| = 2n(n+1)$ and

$$W = (\mathbb{Z}/2\mathbb{Z})^{n-1} \ltimes S_n,$$

where $(\mathbb{Z}/2)^{n-1}$ is the subgroup with even number of sign changes.
Exercise: As before, check all these statements. But D_n is only irreducible if $n \geq 3$. We have the identities $R_{D_3} = R_{A_3}, R_{D_2} = R_{A_1} \coprod R_{A_1}$. These are the root systems of $\mathfrak{so}_{2n}$.

(iv) E_8 : Let

$$\Gamma_n := \{(k_1, ..., k_n) \mid \sum k_i \in 2\mathbb{Z} \text{ and either all } k_i \text{ in } \mathbb{Z} \text{ or all in } \mathbb{Z} + \tfrac{1}{2}\}.$$

Consider $\alpha = (\frac{1}{2}, ..., \frac{1}{2})$, then $(\alpha, \alpha) = \frac{n}{4}$, using the usual inner product. If $\alpha \in \Gamma_n$ and Γ_n is an even lattice, then 8 divides n.
Exercise:

(a) Γ_{8n} is a lattice.

(b) If $n > 1$, the roots of Γ_{8n} are a root system of type D_{8n}.

(c) $R_{\Gamma_8} = \{\pm e_i \pm e_j, i < j, \frac{1}{2}(\pm e_1 \pm ... \pm e_8),$ with even number of minus signs$\}$ is a root system, the root system of type E_8. Note, $|R_{\Gamma_8}| = \frac{8 \cdot 7}{2} 4 + 128 = 240$.

(d) Can you compute $|W|$? (It is $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$).

Remark: A Lie algebra with root system R_{Γ_8} should have dimension $8 + 240 = 248$, as $\dim \mathfrak{t} = \dim_{\mathbb{R}} \mathbb{R}R$, and every root space has dimension 1. The smallest non-trivial representation of such a Lie algebra would also have dimension 248 (adjoint representation).

Exercise 5.16: If R is a root system, $\alpha \in R$, then $\alpha^{\perp} \cap R$ is a root system.

Definition 5.17: We can apply this to R_{Γ_8}. Take $\alpha = \frac{1}{2}(1, ..., 1)$, $\beta = e_7 + e_8$:

(i) $\alpha^{\perp} \cap R_{\Gamma_8}$ is a root system, the root system of type E_7.

(ii) $\alpha^{\perp} \cap \beta^{\perp} \cap R_{\Gamma_8} = \langle \alpha, \beta \rangle^{\perp} \cap R_{\Gamma_8}$ is a root system, the root system of type E_6.

Exercise 5.18: Show $|R_{E_7}| = 126$, $|R_{E_6}| = 72$ and describe the corresponding lattices.

Theorem 5.19:

(i) *"ADE" classification*: The complete list of irreducible simply laced root systems is

$$A_n, n \geq 1, \quad D_n, n \geq 4, \quad E_6, E_7, E_8,$$

and no two root systems in this list are isomorphic.

(ii) The remaining irreducible (reduced) root systems are denoted by

$$B_2 = C_2, \quad B_n, C_n, n \geq 3, \quad F_4, \quad G_2,$$

where

$$\begin{aligned} R_{B_n} &= \{\pm e_i, \pm e_i \pm e_j, i > j\} \subseteq \mathbb{Z}^n \text{ (root system of } \mathfrak{so}_{2n+1}), \\ R_{C_n} &= \{\pm 2e_i, \pm e_i \pm e_j, i > j\} \subseteq \mathbb{Z}^n \text{ (root system of } \mathfrak{sp}_{2n}), \\ R_{C_n}^{\vee} &= R_{B_n}, \\ W_{B_n} &= W_{C_n} = (\mathbb{Z}/2\mathbb{Z})^n \ltimes S_n. \end{aligned}$$

F_4: Put

$$\begin{aligned} Q_n &= \{(k_1, ..., k_n) \mid \forall i, k_i \in \mathbb{Z} \text{ or } \forall i, k_i \in \mathbb{Z} + \frac{1}{2}\}, \text{ and define} \\ R_{F_4} &= \{\alpha \in Q_n \mid (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 1\} \\ &= \{\pm e_i, \pm e_i \pm e_j, i > j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}. \end{aligned}$$

G_2: Consider the lattice $L = \{(x, y, z \in \mathbb{Z}^3 \mid x + y + z = 0)\}$, and define

$$\begin{aligned} R_{G_2} =& \{\alpha \in L \mid (\alpha, \alpha) = 2 \text{ or } 6\} \\ =& \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \\ & \pm(2e_1 - e_2 - e_3), \pm(-e_1 + 2e_2 - e_3), \pm(-e_1 - e_2 + 2e_3)\}. \end{aligned}$$

Exercise 5.20: Check that F_4 as defined in Theorem 5.19 is a root system.

We want to choose a "good" basis for V. Assume, we have $f\colon V \to \mathbb{R}$ linear, s.t. $f(\alpha) \neq 0$, for all $\alpha \in R$. Define $\alpha \in R$ *positive* if $f(\alpha) \geq 0$, and *negative* if $f(\alpha) < 0$. Denote

$$R^+ := \{\alpha \in R \mid f(\alpha) > 0\}, \text{ and } R^- := -R^+.$$

Definition 5.21: A root $\alpha \in R^+$ is *simple* if it is not the sum of two positive roots, i.e. $\alpha \neq \beta + \gamma$, for all $\beta, \gamma \in R^+$. Write $\Pi = \{\alpha_1, ..., \alpha_l\}$ for the set of simple roots. Note that using a different function f may give other simple roots.

Example 5.22:

A_n : Here, $R = \{e_i - e_j \mid i \neq j\}$. Choose $f(e_1) = n + 1, f(e_2) = n, ..., f(e_{n+1}) = 1$, so $R^+ = \{e_i - e_j \mid i < j\}$. $f(R^+) \subset \mathbb{N}$, so if $f(\alpha) = 1$, α must be simple, thus $\Pi = \{e_1 - e_2, e_2 - e_3, ..., e_n - e_{n+1}\}$.

B_n : $R = \{\pm e_i, \pm e_i \pm e_j \mid i < j\}$. Put $f(e_1) = n, ..., f(e_n) = 1$, then $R^+ = \{e_i, e_i \pm e_j \mid i < j\}$ and $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, e_n\}$.

C_n : $R = \{\pm 2e_i, \pm e_i \pm e_j \mid i < j\}$. Using the same f as for B_n, we obtain $R^+ = \{2e_i, e_i \pm e_j \mid i < j\}$ and $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, 2e_n\}$.

D_n : $R = \{\pm e_i \pm e_j \mid i < j\}$. Using the same f as for B_n, C_n, we obtain $R^+ = \{e_i \pm e_j \mid i < j\}$ and $\Pi = \{e_1 - e_2, ..., e_{n-1} - e_n, e_{n-1} + e_n\}$.

E_8 : Consider E_8 with $f(e_1) = 28, f(e_i) = 9 - i, i = 2, ..., 8$ (note $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$), then

$$R^+ = \{e_i \pm e_j (i < j), \frac{1}{2}(e_1 \pm e_2 \pm ... \pm e_8) \text{ (with even number of minus signs)}\},$$
$$\Pi = \{\underbrace{e_2 - e_3, ..., e_7 - e_8}_{f=1}; \underbrace{\frac{1}{2}(e_1 + e_8 - e_2 - ... - e_7)}_{f=2}; \underbrace{e_7 + e_8}_{f=3}\}.$$

Exercise 5.23: Check all theres examples, pick nice functions f and also do E_6, E_7, F_4, G_2.

Proposition 5.24 (Dynkin):

(i) If $\alpha, \beta \in \Pi$, then $\alpha - \beta \notin R$.

(ii) If $\alpha, \beta \in \Pi, \alpha \neq \beta$, then $(\alpha, \beta^\vee) \leq 0$.

(iii) Every $\alpha \in R^+$ can be written as $\alpha = \sum k_i \alpha_i$, with $\alpha_i \in \Pi$ and $k_i \in \mathbb{Z}_{\geq 0}$.

(iv) Simple roots are linearly independent (i.e. the sum in (iii) is unique). Remark: This shows that Π is the desired "nice" basis for V.

(v) If $\alpha \in R^+ \setminus \Pi$, then there exist $\beta \in \Pi$, s.t. $\alpha - \beta \in R^+$.

(vi) R irreducible $\iff$ Π is indecomposable, i.e. $\Pi \neq \Pi_1 \coprod \Pi_2$ with $(\Pi_1, \Pi_2) = 0$.

Proof: Exercise. Either case by case checking, or finding an uniform proof from the axioms of a root system (see e.g. (Kac, 2010, Thm 17.1)). □

Definition 5.25: Let $\Pi = \{\alpha_1, ..., \alpha_l\}$. Define $a_{ij} = (\alpha_i, \alpha_j^\vee)$, $A = (a_{ij})_{1 \leq i,j \leq l}$ is the *Cartan matrix.*

Proposition 5.26: The Cartan matrix $A = (a_{ij})$ of a root system satisfies the following properties:

(i) $a_{ij} \in \mathbb{Z}$, for all i, j, $a_{ii} = 2$, $a_{ij} \leq 0$ if $i \neq j$;

(ii) $a_{ij} = 0 \iff a_{ji} = 0$;

(iii) $\det A > 0$;

(iv) all principal subdeterminants of A have positive determinant.

Proof: (i), (ii) have been proven before.
(iii):

$$A = \begin{pmatrix} \frac{2}{(\alpha_1,\alpha_1)} & & 0 \\ & \ddots & \\ & & \frac{2}{(\alpha_l,\alpha_l)} \end{pmatrix} ((\alpha_i, \alpha_j)),$$

where $\det((\alpha_i, \alpha_j)) > 0$ as it is the Gram matrix of a positive definite bilinear form. For (iv) notice that the principal subdeterminants are matrices of exactly the same form, thus also have positive determinant (or argue that the restriction of the bilinear form to $\langle \alpha_1, ...\alpha_{l-k} \rangle$, $k = 0, ..., l-1$, is also positive definite). □

We can draw A as a graph using so-called *Dynkin diagrams.* In these diagrams, vertices are simple roots, and edges are given by $a_{ij}a_{ji}$ lines joining simple roots α_i and α_j. Note that for irreducible root systems on the following values appear:

$$a_{ij}a_{ji} = \begin{cases} 1, & \text{if simply laced,} \\ 2, & \text{appears in } B_n,\ C_n,\ F_4 \\ 3, & \text{appears in } G_2. \end{cases}$$

If $a_{ij}a_{ji} = 2$ or 3 put an arrow in the direction of the short root. The Dynkin diagrams of all the root systems (classification in Theorem 5.19) are shown in Figure 2.

Exercise 5.27: Show that the Dynkin diagrams are as claimed in Figure 2.

Exercise 5.28: If (R, V) is an irreducible root system with positive roots R^+ and simple roots Π, then there exists an unique positive root $\theta \in R^+$, s.t. for all $\alpha_i \in \Pi$ $\theta + \alpha_i \notin R$. θ is called the *highest root.* Note, as $s_\alpha \theta \in R, (\alpha_i, \theta) < 0, \forall i$.

Solution: Examine the roots systems one by one (later, we will give a uniform proof of this statement). E.g. for A_n, take $\theta := e_1 - e_{n+1}$. □

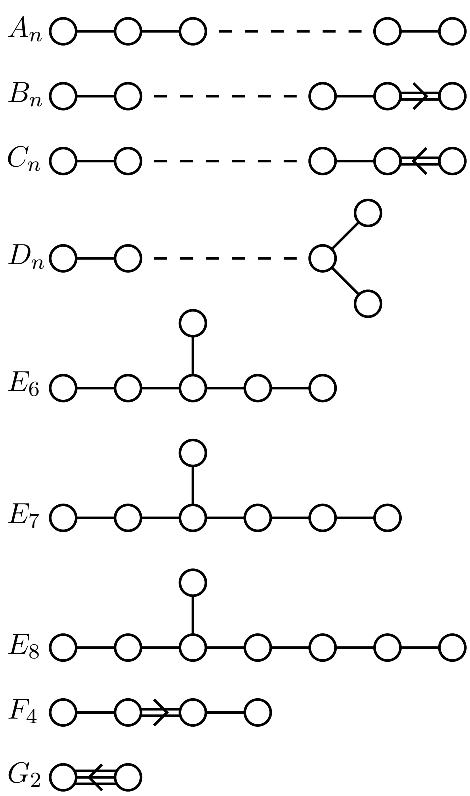


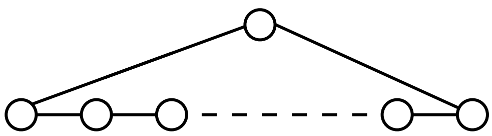
Figure 2: Dynkin diagrams

Define the *extended Cartan matrix* $\tilde{A}$ by setting $\alpha_0 = -\theta$, $\tilde{A} = (a_{ij})_{0\le i,j\le l}$, where $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = (\alpha_i, \alpha_j^\vee)$.

Example 5.29:

A_1 : $A = (2)$, take $\theta = \alpha$, as this is the only positive root, then $\tilde{A} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$.

A_n : The extended Cartan matrix is $\tilde{A}_n = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$, if $n > 1$. The Dynkin diagram of $\tilde{A}_n$ is



Corollary 5.30: Notice that $\tilde{A}$ satisfies:

(i) $a_{ij} \in \mathbb{Z}$, for all i, j, $a_{ij} \le 0$ if $i \ne j$;

(ii) $a_{ij} = 0 \iff a_{ji} = 0$;

(iii) $\det \tilde{A} = 0$, and all principal subdeterminants A of $\tilde{A}$ have $\det A > 0$.

Proof: (i) and (ii) follow directly from the properties of A. For (iii) notice that $\Pi \cup \{\theta\}$ is *not* linearly independent. $\square$

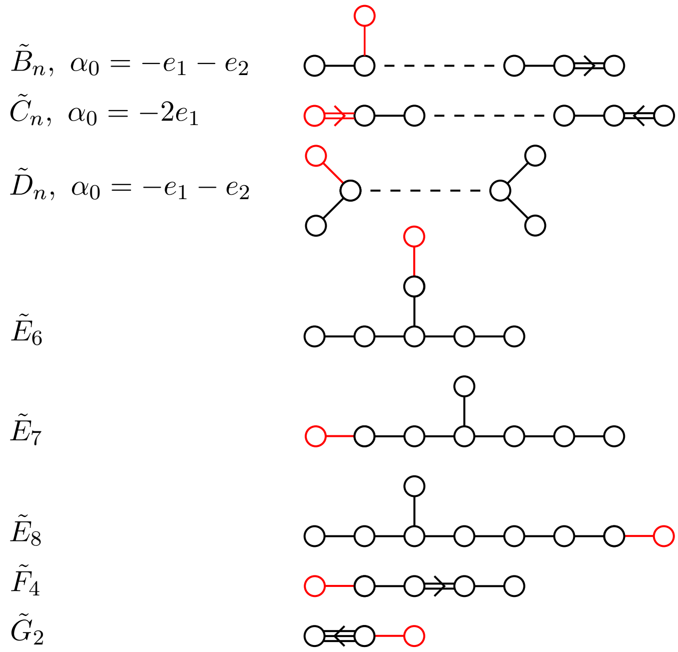


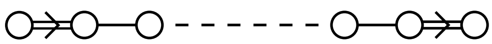
Figure 3: Extended Dynkin diagrams $\tilde{A}$

Exercise 5.31: Write down the highest root θ and the extended Dynkin diagram $\tilde{A}$ for all types of root systems.

Solution: See Figure 3. $\square$

Exercise 5.32:

(i) Show the corresponding Dynkin matrix to



also has determinant 0. We call this matrix *twisted* $\tilde{A}_n$, denoted by $\tilde{A}_n^{(2)}$.

(ii) The Dynkin diagram of A^T is the Dynkin diagram of A with the arrows reversed.

Theorem 5.33: An irreducible (i.e. connected) Dynkin diagram, and hence an indecomposable Cartan matrix is one of $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

Proof:

(i) Classify the rank 2 Dynkin diagrams. These have a Cartan matrix of the form

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix} \implies \det A = 4 - ab > 0$$

$$ab = \underbrace{(0,0)}_{A_1 \times A_1}, \underbrace{(1,1)}_{A_2}, \underbrace{(2,1),(1,2)}_{B_2}, \underbrace{(3,1),(1,3)}_{G_2,}$$

are the only possibilities, as $a_{ij}a_{ji} \in \{0,1,2,3\}$.

(ii) Observe that any subdiagram of a Dynkin diagram is a Dynkin diagram (follows from the fact that the principal subminors have determinant > 0).

(iii) Dynkin diagrams contain no cycles. To prove this, let $\alpha_1,...,\alpha_n$ be distinct simple roots and consider

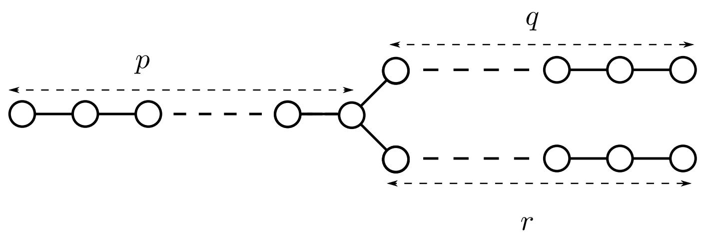
$$\begin{aligned}
\alpha &= \sum_{i=1}^{n} \frac{\alpha_i}{\sqrt{(\alpha_i,\alpha_i)}} \in V, \text{ then} \\
0 < (\alpha,\alpha) &= n + \sum_{i<j} \frac{2(\alpha_i,\alpha_j)}{\sqrt{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}} \\
&= n - \sum_{i<j} \sqrt{a_{ij}a_{ji}},
\end{aligned}$$

by definition of $a_{ij} = (\alpha_i, \alpha_j^\vee)$ and the fact that $(\alpha_i,\alpha_j^\vee) \leq 0$ if $i \neq j$.
So $\sum_{i<j} \sqrt{a_{ij}a_{ji}} < n$. But now if there is a cycle on $\alpha_1, ...\alpha_n$, we must have n or more edges, i.e. $\sum_{i<j} \sqrt{a_{ij}a_{ji}} \geq n$, a contradiction.

(iv) The Dynkin diagram does not contain any extended Dynkin diagrams.

(v) If the diagram is simply laced (i.e. no 2 or 3 bonds), then it is of type A, D or E. To prove this, suppose such a diagram is not of type A, D or E. As $\tilde{D}_4$ is not contained in any Dynkin diagram, we only have triple branch points. Denote by $T_{p,q,r}$ the diagram

$T_{p,q,r}$

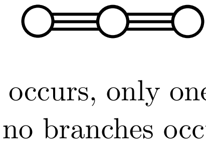


having 3 branches with p, q, and r edges (e.g. $E_8 = T_{5,3,2}$). Exercise: finish the proof by

(a) arguing that, as a Dynkin diagram does not contain $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, we are left with D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$,

(b) showing that $\det T_{p,q,r} = pq+pr+qr-pqr$ by induction on $p+q+r$, and hence argue that D_n, E_6, E_7, E_8 are the only possibilities for this. E.g. $\det E_8 = 15+10+6-30 = 1$.

(vi) Consider the case if the diagram is not simply laced.
Exercise: If G_2 is a subdiagram, then the diagram is G_2. Hint: We have seen



does not appear, show

also do not appear.

Finally, if a 2 bond occurs, only one of such as $\tilde{C}_n$ and $\tilde{A}_n^{(2)}$ are not contained in a Dynkin diagram, and then no branches occur as $\tilde{B}_n$ is not contained. If the double bond is in the middle, the diagram has to be F_4 (otherwise, it contains $\tilde{F}_4$). If the double bound is not in the middle, we have B_n or C_n. $\square$

Exercise 5.34: Compute the determinant of all the Cartan matrices. For example:

A_1: $A_1 = (2)$ has $\det A_1 = 2$,

A_2: $A_2 = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ has $\det A_2 = 3$, etc.

Remark 5.35: Notice that $SL_{n+1} = \{X \mid \det X = 1\}$ has center isomorphic to the cyclic group of order $n+1$. The order of this is the determinant of the Cartan matrix. In general, the order of the center of the simply connected group with Lie algebra $\mathfrak{g}$ whose Cartan matrix is A is $\det A$.

6 Existence and Uniqueness

Setting: For a semisimple Lie algebra $\mathfrak{g}$, we chose a maximal torus $\mathfrak{t}$ and obtained a direct sum composition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Further, we chose $f \colon \mathbb{R}R \to \mathbb{R}$, giving us R^+, and thus the simple roots $\Pi = \{\alpha_1, \dots, \alpha_l\}$. From this we defined the Cartan matrix A.

(A) Independence of choices

Theorem 6.1: Let $\operatorname{char} k = 0$ and $k = \overline{k}$. All maximal tori are conjugate, i.e. if $\mathfrak{t}$ and $\mathfrak{t}'$ are maximal tori of $\mathfrak{g}$, then there exists some $g \in (\operatorname{Aut}\mathfrak{g})^\circ = \{g \in GL(\mathfrak{g}) \mid g \colon \mathfrak{g} \to \mathfrak{g}$ is a Lie alg. homomorphism$\}^\circ$, such that $g\mathfrak{t} = \mathfrak{t}'$. Note that $\operatorname{Aut}(\mathfrak{g})$ is an algebraic group (with $\operatorname{Lie}(\operatorname{Aut}(\mathfrak{g})) = \mathfrak{g}$). $\operatorname{Aut}(\mathfrak{g})^\circ$ is defined as the connected component which contains the $1 \in \operatorname{Aut}(\mathfrak{g})$.

Theorem 6.2: All choices of positive roots R^+ are conjugate. Let (R, V) be a root system. For $f_1, f_2 : V \to \mathbb{R}$ (s.t. $f_i(\alpha) \neq 0\ \forall \alpha \in R$) denote the corresponding sets of positive roots by R_1^+, R_2^+. Then there exists a unique $w \in W$ (the Weyl group), such that $wR_1^+ = R_2^+$. Hence $w\Pi_1 = \Pi_2$ and thus they have the same Cartan matrix.

Corollary 6.3: $\mathfrak{g}$ determines the Cartan matrix, regardless of the choices of the maximal torus and the function f.

(B) Uniqueness

Theorem 6.4: Let $\mathfrak{g}_i$ for $i = 1, 2$ be semisimple Lie algebras with respective $\mathfrak{t}_i, R_i, R_i^+, \Pi_i, A_i$. Assume that after reordering of indices, we have $A_1 = A_2$. Then there exists some isomorphism $\varphi : \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{g}_2$, such that $\varphi(\mathfrak{t}_1) = \mathfrak{t}_2$, $\varphi(R_1) = R_2$, etc.

(C) Existence

Theorem 6.5: Let A be a Cartan matrix. Then there exists a semisimple Lie algebra with A as its Cartan matrix.

Remark 6.6: We already know this, except for G_2, F_4, E_6, E_7, E_8.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Choose some non-zero $E_i \in \mathfrak{g}_{\alpha_i}$ and $F_i \in \mathfrak{g}_{-\alpha_i}$, such that $(E_i, F_i)_{\mathrm{ad}} = \frac{2}{(\alpha_i,\alpha_i)}$ (this is possible as $(\cdot,\cdot)|_{\mathfrak{g}_\alpha}$ is non-degenerate (4.11) and $(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha) = 0$) and let

$$H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathfrak{t}, \quad a_{ij} = (\alpha_i^\vee, \alpha_j) = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}.$$

We have $[H_i, H_j] = 0$ (since the H_i are in the torus, thus commuting), $[E_i, F_i] = H_i$, $[E_i, F_j] = 0$ if $i \neq j$ (since $[E_i, F_i] \in \mathfrak{g}_{\alpha_i - \alpha_j}$, which is no root for $i \neq j$, making $\mathfrak{g}_{\alpha_i-\alpha_j} = 0$) and

$$[H_i, E_j] = \alpha_j(H_i)E_j = \frac{\alpha_j(2\nu^{-1}(\alpha_i))}{(\alpha_i,\alpha_i)} \cdot E_j = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} \cdot E_j = a_{ij}E_j \quad \text{and so} \quad [H_i, F_j] = -a_{ij}F_j$$

Let $\mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in R^-} \mathfrak{g}_\alpha$. So $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{t} \oplus \mathfrak{n}^-$.

Lemma 6.7: The E_i generate $\mathfrak{n}^+$ and the F_i generate $\mathfrak{n}^-$. Hence, $\{E_i, F_i\}$ generates $\mathfrak{g}$ (as a Lie algebra).

Proof: Let $\alpha = \sum k_i\alpha_i \in R^+$, so $k_i \geq 0$. Define the *height* of α as $\mathrm{ht}(\alpha) = \sum k_i \geq 0$. Induct on $\mathrm{ht}(\alpha)$ that $\mathfrak{g}_\alpha$ is spanned by linear combinations of the E_i. If $\mathrm{ht}(\alpha) = 1$, then $\alpha = \alpha_i \in \Pi$ for some i, so $\mathfrak{g}_{\alpha_i} = \mathbb{C}E_i$. If $\mathrm{ht}(\alpha) > 1$ we know that there exists some $\alpha_i \in \Pi$, such that $\beta = \alpha - \alpha_i \in R^+$ (by 5.24(v)). But we know $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta] = \mathfrak{g}_\alpha$ by structure theorem, as α_i, α, β are all roots. So by induction hypothesis, $\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta$ are generated by some E_i, and thus also $\mathfrak{g}_\alpha$. Inductively, this proves that the E_i generate $\mathfrak{n}^+$ as a Lie algebra. The argument for $\mathfrak{n}^- = \langle F_i \rangle$ is similar. Finally $[E_i, F_i] = H_i$. But $\mathfrak{t} = \langle H_i : i = 1, \dots, l\rangle$. So E_i, F_i generate $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{t} + \mathfrak{n}^-$. $\square$

Now let A be a *generalized Cartan matrix*, i.e. $a_{ii} = 2$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, $a_{ij} \in -\mathbb{N}$ if $i \neq j$.

Definition 6.8:

(i) Let $\widetilde{\mathfrak{g}}$ denote the Lie algebra with generators E_i, F_i, H_i, where $i = 1, \dots, l$, and the relations

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, E_j] &= a_{ij}E_j, \\ [H_i, F_j] &= -a_{ij}F_j, \\ [E_i, F_j] &= 0, \text{ if } i \neq j, \\ [E_i, F_i] &= H_i, \end{aligned}$$

as above. (Remark: so $\widetilde{\mathfrak{g}}$ is basically a "bunch of $\mathfrak{sl}_2$ glued together.")

(ii) Let $\overline{\mathfrak{g}}$ be the quotient of $\widetilde{\mathfrak{g}}$ by the additional relations

$$(\operatorname{ad} E_i)^{1-a_{ij}}E_j = 0 \quad \text{and} \quad (\operatorname{ad} F_j)^{1-a_{ij}}F_j = 0, \text{ if } i \neq j,$$

the so-called *Serre relations* (though discovered by Harish-Chandra, Chevalley). (Note that if $a_{ij} = 0$, then these relations become $[E_i, E_j] = 0$; if $a_{ij} = -1$, $[E_i, [E_i, E_j]] = 0$.)

Exercise 6.9: Check that the Serre relations hold for the classical groups $\mathfrak{sl}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$.

Theorem 6.10:

(i) If A is indecomposable, then $\widetilde{\mathfrak{g}}$ has a unique maximal ideal and $\overline{\mathfrak{g}}$ is its quotient, i.e. $\overline{\mathfrak{g}}$ is simple (not necessarily finite-dimensional).

(ii) Hence, if $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra with Cartan matrix A, then the map $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ ($E_i \mapsto E_i$, $F_i \mapsto F_i$) factors through $\overline{\mathfrak{g}}$, is surjective and gives an isomorphism $\overline{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$.

Remark: (i)⇒(ii) follows from Lemma 6.7, (ii) implies uniqueness as stated in (6.4).

The above theorem shows that existence is equivalent to the following theorem:

Theorem 6.11: $\overline{\mathfrak{g}}$ is finite dimensional if and only if A is a Cartan matrix.

Definition 6.12: In general, $\overline{\mathfrak{g}}$ is called a *Kac-Moody algebra*.

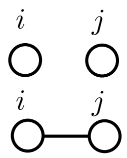
Theorem 6.13 (Presentation of W): Write $r_i = s_{\alpha_i}$, then

$$W = \langle r_1, \ldots, r_l \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$$

is a presentation of W, where

$a_{ji}a_{ij}$	0	1	2	3
m_{ij}	2	3	4	6

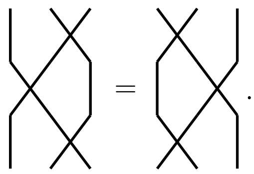
Example 6.14: Consider the simple cases



$r_i r_j = r_j r_i$

$r_i r_j r_i = r_j r_i r_j$

For A_n, we have the relations $r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ and $r_i r_j = r_j r_i$ if $j \neq i+1$, visualized as



Exercise 6.15: Check for each root system that the relations claimed do hold. (Hint: it is enough to show this for all rank 2 root systems).

7 Representations of Semisimple Lie Algebras

7.1 Classification of finite-dimensional representations

From now on, let $\mathfrak{g}$ be a semisimple Lie algebra, so

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-.$$

Furthermore, let V be a finite-dimensional representation of $\mathfrak{g}$.

Proposition 7.1: In the above setting, we have:

(i) $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$, where $V_\lambda = \{v \in V : tv = \lambda(t)v \ \forall v \in V\}$, the weight space decomposition w.r.t. $\mathfrak{t}$ (i.e. $\mathfrak{t}$ acts semisimply on V).

(ii) If $V_\lambda \neq 0$, then $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$. (Recall that we had $(\mathfrak{sl}_2)_\alpha = \langle e_\alpha, h_\alpha, e_{-\alpha} \rangle$, $h_\alpha = \nu^{-1}(\alpha^\vee)$, for all $\alpha \in R$).

Proof: As V is a finite-dimensional representation, it is also a finite dimensional representation for $(\mathfrak{sl}_2)_\alpha$, so h_α acts diagonalizable on V by the $\mathfrak{sl}_2$-theory and $\lambda(h_\alpha) \in \mathbb{Z}$. As the h_α span $\mathfrak{t}$, (i) follows immediately. □

Definition 7.2: Let R be a root system with simple roots $\alpha_1, \ldots, \alpha_l$.

(i) Set $Q = \mathbb{Z}R = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i$, the *lattice of roots* of R.

(ii) Set $P = \{\gamma \in \mathbb{Q}R \mid \forall \alpha \in R : (\gamma, \alpha^\vee) \in \mathbb{Z}\} = \{\gamma \in \mathbb{Q}R \mid \forall i : (\gamma, \alpha_i^\vee) \in \mathbb{Z}\}$, the *lattice of weights* of R.

Remark 7.3: Note, if $\beta, \alpha \in R$, then $(\beta, \alpha^\vee) \in \mathbb{Z}$, so $Q \subseteq P$. Notice also $(\gamma, \alpha^\vee) = \gamma(h_\alpha)$, so if V is a finite dimensional-representation of $\mathfrak{g}$ and $V_\lambda \neq 0$, then $\lambda \in P$ by Proposition (7.1ii).

Exercise 7.4:

(i) Show $|P/Q| < \infty$, in fact $|P/Q| = \det A$, where A is the Cartan matrix of $\mathfrak{g}$.

(ii) Show that the Weyl group W acts on $\mathfrak{t}^*$, and $W \cdot P \subseteq P$, hence W acts on P.

Example 7.5: Consider $\mathfrak{sl}_2$ with $R = \{\pm\alpha\}$. Then $Q = \mathbb{Z}\alpha$. Since $(\alpha, \alpha) = 2$, this means $P = \mathbb{Z}\frac{\alpha}{2}$. So here we have $|P/Q| = 2 = \det(2) = \det A$.

Definition 7.6: If V is a finite-dimensional representation of $\mathfrak{g}$, define the *character* of V

$$\operatorname{ch} V = \sum_{\lambda \in P} \dim V_\lambda e^\lambda \in \mathbb{Z}[P]$$

where e^λ is a formal symbol, basis for $\mathbb{Z}[P]$, with $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

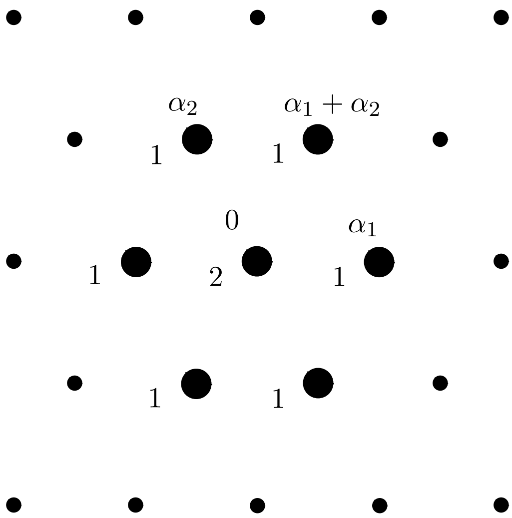
Example 7.7: For $\mathfrak{sl}_2$, we have $P = \mathbb{Z}\frac{\alpha}{2}$. Write $z = e^{\frac{\alpha}{2}}$, then

$$\operatorname{ch} L(n) = z^n + \ldots + z^{-n} = \frac{z^{n+1} + z^{-(n+1)}}{z + z^{-1}}.$$

Now look at the adjoint representation of $V = \mathfrak{g}$ for $\mathfrak{sl}_3$. Put $w = e^{\alpha_1}$ and $z = e^{\alpha_2}$. Then

$$\operatorname{ch} V = 2 + z + w + z^{-1} + w^{-1} + zw + z^{-1}w^{-1},$$

For the root lattice we have the following picture:



where the numbers next to the roots indicate the dimensions of the root spaces. Note that this picture is S_3 invariant.

Proposition 7.8: Let V be a finite-dimensional representation of $\mathfrak{g}$, then $\dim V_\lambda = \dim V_{w\lambda}$ for any $w \in W$, i.e. $\operatorname{ch} V$ is W-invariant ($\operatorname{ch} V \in \mathbb{Z}[P]^W$).

Proof Sketch 1. If G is an algebraic group with $\mathfrak{g} = \operatorname{Lie}(G)$ and T is the subgroup with $\mathfrak{t} = \operatorname{Lie}(T)$ (e.g. $\mathfrak{g} = \mathfrak{sp}_{2n}, G = SP_{2n}$, T are the diagonal matrices in SP_{2n}), then $W = N(T)/T$ (we do not prove this result; for example in $\mathfrak{sl}_n$, T is the set of diagonal matrices, $N(T)$ are the basis matrices E_{ij} (monomial matrices) and $N(T)/T = S_n$), so for any $w \in W$ there exists $\dot{w} \in N(T)$, such that $\dot{w}T = w$. Now if G acts on V (always if G is simply connected), then $\dot{w}(V_\lambda) = V_{w\lambda}$ as $t\dot{w}(v) = \dot{w}w^{-1}tw \cdot v = \dot{w}(\lambda(w^{-1}tw)v) = \lambda(w^{-1}tw)\dot{w}v$. $\square$

Example 7.9: $\dot{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\dot{s}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in T \backslash \{1\}$ shows that you cannot embed $W \hookrightarrow G$ in general and W itself does *not* act on V. Instead we have a small 2-group ($\cong (\mathbb{Z}/2)^l$ at worst), that intervenes. So W is the normalizer of the maximal torus modulo the torus.

Proof Sketch 2. Mimic this in $\mathfrak{g}$. How to see

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \exp(f)\exp(-e)\exp(f)? \quad \left(\text{where} \quad \exp(x) = \sum_{n \geq 0} \frac{x^n}{n!} \right)$$

This is only easy for nilpotent matrices. So for each not α, define

$$\dot{s}_\alpha = \exp(f_\alpha)\exp(-e_\alpha)\exp(f_\alpha).$$

The following steps (*exercise*) finish the proof:

(i) If V is a finite-dimensional representation of $\mathfrak{g}$, then e_α, f_α obviously act nilpotently on V, so $\dot{s}_\alpha : V \to V$ is a well-defined finite sum.

(ii) $\dot{s}_\alpha^2 = \varepsilon_\alpha \colon V_\lambda \to V_\lambda$, where $\varepsilon_\alpha^2 = 1$. ε_α is multiplication by a scalar. Determine it explicitly in terms of λ (denote the scalar by s_α).

(iii) Then $\dot{s}_\alpha V_\lambda = V_{s_\alpha \lambda}$. □

Remark 7.10: We do not need V to be finite-dimensional for this, just need that each e_α, f_α acts locally nilpotent (some $x \colon V \to V$ acts locally nilpotent if for all $v \in V$ there exists $N \in \mathbb{N}$ s.t. $x^N v = 0$).

Exercise 7.11: Show that this is equivalent to V splitting up – as an $(\mathfrak{sl}_2)_\alpha$-module – into a direct sum (possibly infinite) of finite-dimensional $(\mathfrak{sl}_2)_\alpha$-modules, for all $\alpha \in \Pi$. Such a V is called *integrable*.

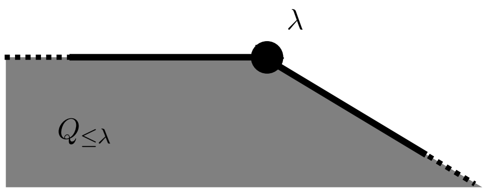
In the following, all the statements for Lie algebras and their proofs also work in the case of Kac-Moody algebras if whenever the assumption of finite dimension of V is made, this is replaced by the condition of V being integrable.

Proof Sketch 3. The statement is actually obvious from the $\mathfrak{sl}_2$-theory: Consider V as a representation of $(\mathfrak{sl}_2)_\alpha \times \mathfrak{t}$, then V breaks up into a direct sum of strings, each of which is of the form

$$\lambda, \lambda - \alpha, \ldots, \lambda - m\alpha,$$

where $m = \lambda(h_\alpha)$. Such a string is obviously s_α invariant. □

Definition 7.12: For $\mu, \lambda \in \mathfrak{t}^*$ write $\mu \leq \lambda$ to mean $\lambda - \mu = \sum k_i \alpha_i, k_i \in \mathbb{N}$. Graphically, this means that $Q_{\leq \lambda} = \{\mu \in P \mid \mu \leq \lambda\}$ is the set of lattice points in an obtuse cone.



Definition 7.13: Let V be a representation of $\mathfrak{g}$, we say

(i) The *weight* of a vector $0 \neq v \in V$ is defined as λ if $v \in V_\lambda$, write $\mathrm{wt}(v) = \lambda$ in this case.

(ii) $\lambda \in P$ is a *highest weight* if $V_\lambda \neq 0$ (i.e. λ is a *weight*) and if $V_\mu \neq 0$, then $\mu \leq \lambda$.

(iii) Say $v \in V_\gamma$ is a *singular vector* if $v \neq 0$ and $e_\alpha v = 0$ for all $\alpha \in R^+$. Note that $\mathrm{wt}(e_\alpha v) = \alpha + \beta > \beta$, if $e_\alpha v \neq 0$. (This follows from $\mathfrak{g}_\alpha V_\lambda \subset V_{\lambda+\alpha}$, as for $x \in \mathfrak{g}$ we have $h_\beta x v = ([h_\beta, x] + x h_\beta) v = (\alpha(h_\beta) + \lambda(h_\beta)) v)$. So if γ is a highest weight, all $0 \neq v \in V_\gamma$ are singular vectors.

(iv) A weight μ is an *extremal weight* if $w\mu$ is a highest weight for some $w \in W$.

(v) Set

$$\begin{aligned} P^+ &= \{\lambda \in P \mid (\lambda, \alpha^\vee) \geq 0, \forall \alpha \in R^+\} \\ &= \{\lambda \in P \mid (\lambda, \alpha_i^\vee) \geq 0, \forall \alpha_i \in \Pi\} \end{aligned}$$

and call P^+ the *cone of dominant weights.*

In the picture for the $\mathfrak{sl}_3$ root lattice, all points of the outside hexagon are extremal weights, and $\alpha_1 + \alpha_2$ is highest weight. Note that if V is finitely-dimensional, then highest weights exist, what implies that singular vectors exist.

Theorem 7.14: Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$.

(A) (**Complete reducibility**) If V is a finite-dimensional representation of $\mathfrak{g}$, then V is a direct sum of irreducibles.

We have $P^+ \cong \{\text{irreducible f.-d. representations of } \mathfrak{g}\}$ via $\lambda \mapsto L(\lambda)$. More precisely:

(B) Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$, $v \in V_\lambda$ a singular vector, then:

(i) $V_\lambda = \mathbb{C} \cdot v$, i.e. $\dim V_\lambda = 1$.

(ii) If $V_\mu \neq 0$, then $\lambda \leq \mu$, so v is a highest weight vector (we say V *has highest weight* λ).

(iii) $\lambda(h_i) \in \mathbb{N}$ for all $i = 1, \ldots, l$, i.e. $\lambda \in P^+$.

Moreover, if U is another irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ, and $u \in U_\lambda$, then there exists a unique isomorphism $V \to W$ sending $v \mapsto w$.

(C) Given $\lambda \in P^+$, there exists a finite-dimensional irreducible representation with highest weight λ, denoted by $L(\lambda)$

(D) We will later give a closed formula for $\operatorname{ch} L(\lambda)$, the so-called *Weyl character formula.*

Corollary 7.15: $\operatorname{ch} L(\lambda) = e^\lambda + \sum_{\mu<\lambda} a_\mu e^\mu \in \mathbb{Z}[P]$, and hence $\{\operatorname{ch} L(\lambda) \mid \lambda \in P^+\}$ are linearly independent. Write $\operatorname{ch} L(\lambda) = m_\lambda + \sum_{\mu<\lambda} \tilde{a}_{\mu\lambda} m_\mu \in \mathbb{Z}[P]$, where $m_\mu = \sum_{\gamma \in W} e^{\gamma\mu}$, the so-called *monomial symmetric functions.* As the m_μ clearly form a basis of $\mathbb{Z}[P]^W$, this shows that $\operatorname{ch}\{L(\lambda)\}$ is a basis of $\mathbb{Z}[P]^W$.

Corollary 7.16: If V, W are finite-dimensional, then $V \cong W$ if and only if $\operatorname{ch} V = \operatorname{ch} W$.

Proof: Apply complete reducibility and the previous corollary. □

Remark 7.17: Define $w_i \in P$ to be the dual basis to the simple coroots h_{α_i}, i.e. $(w_i, \alpha_j^\vee) = \delta_{ij}$, for $i = 1, \ldots, l$. These w_i are called the *fundamental weights.* Using this notion, we can write

$$P^+ = \bigoplus_{i=1}^{l} \mathbb{Z}_{\geq 0} w_i = \left\{ \sum_{i=1}^{l} n_i w_i \;\middle|\; n_i \leq 0 \right\}.$$

Exercise 7.18:

(i) Compute P^+ for $\mathfrak{sl}_n, \mathfrak{so}_{2n}, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \ldots$ and draw the picture for A_2, B_2, G_2.

(ii) If $\lambda \in P$, then $\lambda = \sum_i \lambda(h_i) w_i$, where $h_i = \nu^{-1}(\alpha_i^\vee)$ as always.

Example 7.19:

(i) For any Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we have the trivial representation $\mathbb{C} = L(0)$.

(ii) For $\mathfrak{g}$ we have the adjoint representation $\mathfrak{g}$ as a representation of itself, $\mathfrak{g} = \mathfrak{t} + \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. Here, a highest weight λ is a root s.t. $\lambda + \alpha_i \notin R$, for all i. Then $\lambda = \theta$ is the highest root in R^+. Now Theorem 7.14 implies that θ is unique as promised (as $\mathfrak{g}$ is simple if and only if $\operatorname{ad}\mathfrak{g}$ is irreducible).
Take e.g. A_{n-1} as an concrete example. Then $\theta = \varepsilon_1 - \varepsilon_n$, $h_i = E_{ii} - E - i + 1, i + 1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, so $\theta(h_1) = 1, \theta(h_2) = 0, \ldots, \theta(h_{n-2}) = 0, \theta(h_{n-1}) = 1$.
Exercise: Compute $\theta(h_i)$ for all simple Lie algebras.

(iii) Examples of representations of $\mathfrak{sl}_n$. $P = \mathbb{Z}^n/\mathbb{Z} \cong \mathbb{Z}^{n-1}$. Take $\mathbb{C}^n$ as standard representation, with basis $v_1, \ldots, v_n$ and weights $e_1, \ldots, e_n$, $\sum e_i = 0$. Then the highest weight is e_1 (as $e_1 > e_2 > \cdots > e_n$ since $e_1 = (e_1 - e_2) + e_2$, etc.) Then $L(w_1) = \mathbb{C}^n$, $\operatorname{ch}\mathbb{C}^n = e^{e_1} + \ldots + e^{e_n}$. If we write $z_i = e^{e_i}$, this becomes $\operatorname{ch}\mathbb{C}^n = z_1 + \ldots + z_n$, and $\mathbb{Z}[P] = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/(z_1 \cdot \ldots \cdot z_n = 1)$.

Recall that if V, W are representations of $\mathfrak{g}$, then so is $V \otimes W$, where $x \in \mathfrak{g}$ acts by $x \otimes 1 + 1 \otimes x$. Hence, $V \otimes V$ is a representation, but $\sigma \colon V \otimes V \to V \otimes V$, $a \otimes b \mapsto b \otimes a$, commutes with the $\mathfrak{g}$-action, so its eigenspaces are $\mathfrak{g}$-modules. $\sigma^2 = 1$, so the only eigenvalues are ± 1, i.e. S^2V (the symmetric algebra $V \otimes V / \langle v \otimes w - w \otimes v \mid v, w \in V\rangle$ with product $vw := \frac{1}{2}(v \otimes w + w \otimes v)$) and $\Lambda^2 V$ (the exterior algebra $V \otimes V/\langle v \otimes w + w \otimes v \mid v, w \in V\rangle$, with product $v \wedge w := \frac{1}{2}(v \otimes w - w \otimes v)$) are $\mathfrak{g}$-modules. In general, these must not be irreducible, but for $\mathfrak{sl}_n$ they are.

Example 7.20: Let $V = \mathbb{C}^n$ as $\mathfrak{sl}_n$-module as above.

(i) Consider $\Lambda^s V$, $s \leq n-1$, this space has a basis $\{v_{i_1} \wedge \ldots \wedge v_{i_s} \mid i_1 < \cdots < i_s\}$ (if $\{v_1, \ldots, v_n\}$ is a basis of V). Further, $\Lambda^s V$ has weights $e_{i_1} + \ldots + e_{i_s}$ (as $x(w_{i_1} \wedge \ldots \wedge w_{i_s}) = xw_{i_1} \wedge w_{i_2} \ldots \wedge w_{i_s} + \ldots + w_{i_1} \wedge \ldots \wedge w_{i_{s-1}} \wedge xw_{i_s}$), and check that $E_i \cdot (v_{i_1} \wedge \ldots \wedge v_{i_s}) = 0$, for all i, if and only if $v_{i_1} \wedge \ldots \wedge v_{i_s} = v_1 \wedge v_2 \wedge \ldots \wedge v_s$, i.e. this is the only singular vector. So $\Lambda^s \mathbb{C}^n$ is an irreducible $\mathfrak{sl}_n$-module with highest weight $w_s = e_1 + \ldots + e_s$ (the s-th *fundamental weight*), as $(w_s, e_i - e_{i+1}) = \delta_{is}$. Thus, $\Lambda^s \mathbb{C}^n = L(w_s)$. For example, $\Lambda^{n-1}\mathbb{C}^n \cong (\mathbb{C}^n)^* = L(w_{n-1})$.

(ii) Consider $S^m\mathbb{C}^n$, the m-th symmetric power of $\mathbb{C}^n$ with basis $\{v_{i_1} \cdot \ldots \cdot v_{i_m} \mid i_1 \leq \cdots \leq i_m\}$. These are weight vectors with weights $e_{i_1} + \ldots + e_{i_m}$.

$$E_i \cdot (v_{i_1} \cdot \ldots \cdot v_{i_m}) = 0 \ \forall i \iff v_{i_1} \cdot \ldots \cdot v_{i_m} = v_1 \cdot \ldots \cdot v_1 = v_1^m,$$

so $S^m\mathbb{C}^n$ is irreducible and isomorphic to $L(mw_1)$.

Exercise 7.21:

(i) Check all the statements in the above example, compute $\operatorname{ch}\Lambda^s\mathbb{C}^n$, and $\operatorname{ch} S^m\mathbb{C}^n$.

(ii) Find closed formulas for

$$\sum_{m\geq 0} \operatorname{ch} S^m\mathbb{C}^n \cdot q^m, \quad \text{and}$$

$$\sum_{m\geq 0} \operatorname{ch} \Lambda^m\mathbb{C}^n \cdot q^m.$$

Exercise 7.22:

(i) Let V be finite-dimensional $\mathfrak{g}$-module, then $V^* \otimes W \xrightarrow{\sim} \operatorname{Hom}(V, W)$ as $\mathfrak{g}$-modules via $v^* \otimes w \mapsto (u \mapsto w \cdot v^*(u))$ and $\operatorname{Hom}(V, V) \neq 0$ as it contains Id_V. Note that V^* is a $\mathfrak{g}$-representation by defining $(g \cdot f)(v) = -f(g \cdot v)$ for all $f \in V^*$. (This comes from differentiating the group action $(gf)(v) = f(g^{-1}v)$.)

(ii) Show if $V = \mathbb{C}^n$, $\mathfrak{g} = \mathfrak{sl}_n$, then $V \otimes V^* \cong \mathfrak{sl}_n \oplus \mathbb{C}$ (the sum of the adjoint and the trivial representation). In contrast, $V \otimes V \cong S^2V \oplus \Lambda^2V$ (in general).

Exercise 7.23: Let $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2l}$, $2l = n$, $V = \mathbb{C}^n$ as $\mathfrak{g}$-representation in the obvious manner.

(i) Compute the highest weights of V.

(ii) $V \cong V^*$ via the form defining $\mathfrak{g}$, so $V \otimes V$ has at least three summands (since it must have the trivial subrepresentation). Show that it has exactly three summands, describe them and find their highest weights.

In the following, we will prove Theorem 7.14. Let $\mathfrak{g}$ be any Lie algebra with a non-degenerate bilinear form $(\cdot,\cdot)$ (for example, $\mathfrak{g}$ semisimple with the killing form). Let $x_1, \dots, x_N$ be a basis of $\mathfrak{g}$, with $x^1, \dots, x^N$ dual basis, i.e. $(x_i, x^j) = \delta_{ij}$.

Define $\Omega = \sum x_i x^i$, the *Casimir* of $\mathfrak{g}$.

Lemma 7.24: If $x \in \mathfrak{g}$, then $[\Omega, x] = 0$.

Proof: We will give the proof in two different ways: First,

$$\begin{aligned}
[\Omega, x] &= \big[\sum_i x_i x^i, x\big] \\
&= \sum_i x_i[x^i, x] + \sum_i [x_i, x]x^i,
\end{aligned}$$

as $[\cdot, x]$ is a derivation. Now write $[x^i, x] = \sum a_{ij}x^j$, $[x_i, x] = \sum b_{ij}x_j$. But then

$$\begin{aligned}
a_{ij} &= ([x^i, x], x_j) = ([x_j, x^i], x) \\
b_{ij} &= ([x_i, x], x^j) = ([x^j, x_i], x) = -a_{ji},
\end{aligned}$$

using that $(\cdot,\cdot)$ is an invariant form. So $[\Omega, x] = \sum x_i x^j a_{ij} + \sum x_j x^i b_{ij} = 0$.

We can also prove this without coordinates: We have maps of $\mathfrak{g}$-modules $\mathbb{C} \hookrightarrow \operatorname{End}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^*$ via $\lambda \mapsto \lambda \mathrm{Id}$ (i.e. $1 \mapsto \sum x_i \otimes x^i$), and the isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ is implied by the non-degenerate form $(\cdot,\cdot)$. Further, the $\mathfrak{g}$-action on V implies a map of $\mathfrak{g}$-modules $\mathfrak{g} \to \operatorname{End}(V)$ which gives a $\mathfrak{g}$-module map

$$\mathfrak{g} \otimes \mathfrak{g} \to \operatorname{End}(V) \otimes \operatorname{End}(V) \xrightarrow{\text{multiplication}} \operatorname{End}(V).$$

So we have a map of $\mathfrak{g}$-modules $\mathbb{C} \to \operatorname{End}(V)$, which is the statement of the lemma as it maps $1 \mapsto \Omega$ (i.e. Ω generates the trivial submodule of $\operatorname{End}(V)$). $\square$

Now, let $\mathfrak{g}$ be semisimple. Then $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, and let $(\cdot,\cdot) = (\cdot,\cdot)_{\text{ad}}$ be the killing form. Choose a basis $u_1, \ldots, u_l$ of $\mathfrak{t}$, and $0 \neq x_\alpha \in \mathfrak{g}_\alpha$. Denote the dual basis of $\mathfrak{t}$ by $u^1, \ldots, u^l$ and $x^{-\alpha}$ of $\mathfrak{g}_{-\alpha}$, i.e. $(x_\alpha, x^{-\alpha}) = 1$. Normalize x_α so that $(x_\alpha, x_{-\alpha}) = 1$, then $x^{-\alpha} = x_{-\alpha}$ and $[x_\alpha, x_{-\alpha}] = \nu^{-1}(\alpha)$ (note that in general we had that $x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}$ implies $[x,y] = (x,y)\nu^{-1}(\alpha)$). Hence

$$\begin{aligned}\Omega &= \sum u_i u^i + \sum_{\alpha \in R} (x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha) \\ &= \sum u_i u^i + 2 \sum_{\alpha \in R^+} x_\alpha x_{-\alpha} + \sum_{\alpha \in R^+} \nu^{-1}(\alpha).\end{aligned}$$

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha,$$

then we obtain

$$\Omega = \sum u_i u^i + 2\nu^{-1}(\rho) + 2 \sum_{\alpha \in R^+} x_\alpha x_{-\alpha}. \tag{8}$$

Note that this is (up to normalisation) the same Casimir as defined before, in the case of $\mathfrak{sl}_2$.

Lemma 7.25: Let V be a $\mathfrak{g}$-module, $v \in V$ a singular vector with weight λ (i.e. $\mathfrak{n}^+ v = 0$, $tv = \lambda(t)v$). Then $\Omega v = (|\lambda + \rho|^2 - |\rho|^2) \cdot v$.

Proof: Apply (8) to v, $x_\alpha v = 0$ for all $\alpha \in R^+$, so

$$\begin{aligned}\Omega \cdot v &= \left(\sum_{i=1}^{l} \lambda(u_i)\lambda(u^i) + \lambda(2\nu^{-1}(\rho)) \right) \cdot v \\ &= ((\lambda,\lambda) + 2(\lambda,\rho)) \cdot v.\end{aligned}$$

□

Hence, if V is irreducible, Ω acts on V by $(\lambda,\lambda) + 2(\lambda,\rho)$ by Schur's lemma.

7.2 The PBW theorem

Let $\mathfrak{g}$ be any Lie algebra over k, where k is a field.

Definition 7.26: The *universal enveloping algebra* of $\mathfrak{g}$, $U\mathfrak{g}$ is the associative algebra over k generated by $\mathfrak{g}$ and relations $xy - yx = [x,y]$ for all $x, y \in \mathfrak{g}$.

More formally, if V is a vector space over k, then

$$TV = k + V + V \otimes V + V \otimes V \otimes V + \ldots = \bigoplus_{n \geq 0} V^{\otimes n}$$

is the *tensor algebra* over V, the free associative algebra generated by V. Multiplication $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes (n+m)}$ is defined in the obvious way. Let J be the two-sided ideal in $T\mathfrak{g}$ generated by $x \otimes y - y \otimes x - [x,y]$ for $x, y \in \mathfrak{g}$, then $U\mathfrak{g} = T\mathfrak{g}/J$.

Exercise 7.27: An *enveloping algebra* for $\mathfrak{g}$ is a linear map $\iota \colon \mathfrak{g} \to A$, where A is an associative algebra and ι a k-linear map s.t.

$$\iota(x)\iota(y) - \iota(y)\iota(x) = \iota[x,y].$$

For example, for V a representation of $\mathfrak{g}$, $A = \mathrm{End}(V)$, the action map ι is an enveloping algebra. Show that $U\mathfrak{g}$ is initial in the category of enveloping algebras, i.e. the diagram

$$\begin{tikzcd} & U\mathfrak{g} \arrow[d, "\exists!"] \\ \mathfrak{g} \arrow[ur] \arrow[r, "\iota"'] & A \end{tikzcd} \tag{9}$$

commutes.

Note that the Casimir $\Omega \in U\mathfrak{g}$. Indeed, $\Omega \in Z(U\mathfrak{g})$. Observe that $T\mathfrak{g}$ is a graded algebra, but the relations

$$\underbrace{x \otimes y}_{\deg 2} - \underbrace{y \otimes x}_{\deg 2} - \underbrace{[x, y]}_{\deg 1}$$

are not homogeneous, so $U\mathfrak{g}$ is *filtered*: Define $(U\mathfrak{g})_n$ to be the span of elements of degree $\leq n$ of $\mathfrak{g}$. Then $(U\mathfrak{g})_n (U\mathfrak{g})_m \subseteq (U\mathfrak{g})_{n+m}$. In particular, $k \subseteq (U\mathfrak{g})_o$, $k + \mathfrak{g} \subseteq (U\mathfrak{g})_1, \ldots$.

Exercise 7.28: Show that the above statements hold, and that if $x \in (U\mathfrak{g})_n, y \in (U\mathfrak{g})_m$, then $xy - yx = [x, y] \in (U\mathfrak{g})_{n+m-1}$.

Definition 7.29: For a filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ we set

$$\mathrm{gr}\, F = \bigoplus F_i / F_{i-1}$$

and call $\mathrm{gr}\, F$ the *associated graded algebra*.

Theorem 7.30 (PBW Theorem, Poincaré-Birkhoff-Witt):

(i) $\mathrm{gr}\, U\mathfrak{g} = \bigoplus (U\mathfrak{g})_n / (U\mathfrak{g})_{n-1} \xleftarrow{\sim} S\mathfrak{g}$.

(ii) Equivalently, if $x_1, \ldots, x_N$ is a basis of $\mathfrak{g}$, then $\{x_1^{a_1} \cdot \ldots \cdot x_N^{a_N} \mid a_i \in \mathbb{N}\}$ is basis of $U\mathfrak{g}$. In particular, $\mathfrak{g} \hookrightarrow U\mathfrak{g}$.

Exercise 7.31:

(i) Show that the previous exercise ($x \in (U\mathfrak{g})_n, y \in (U\mathfrak{g})_m$, then $xy - yx = [x, y] \in (U\mathfrak{g})_{n+m-1}$) implies that we have a well-defined map $S\mathfrak{g} \to \mathrm{gr}\, U\mathfrak{g}$ extending the map $\mathfrak{g} \to \mathfrak{g}$.

(ii) This map is surjective, i.e. the monomials above span $U\mathfrak{g}$. The content of the PBW Theorem is then to show that this map also injects. We omit the proof of this.

Exercise 7.32: If V is a representation of $\mathfrak{g}$, $v \in V$, then the $\mathfrak{g}$-submodule of V generated by v is just $U\mathfrak{g} \cdot v$ (the image of the map $U\mathfrak{g} \otimes \mathbb{C}v \to V$).

Definition 7.33: A $\mathfrak{g}$-module V is a *highest weight module* for $\mathfrak{g}$ if there exists a singular vector $v \in V$ (i.e. $\mathfrak{n}^+ v = 0$, $t \cdot v = \lambda(t) v$, for all $t \in \mathfrak{t}$ and some $\lambda \in \mathfrak{t}^*$) such that $V = U\mathfrak{g} \cdot v$.

Lemma 7.34: Observe that it follows that $U\mathfrak{n}^- \cdot v = V$.

Proof: The PBW Theorem 7.30 implies that if $x_1, \dots, x_N$ is a basis of $\mathfrak{g}$ then $x_1^{a_1} \dots x_N^{a_N}$ spans $U\mathfrak{g}$. Taking a basis $x_1, \dots, x_r$ for $\mathfrak{n}^-$, $x_{r+1}, \dots, x_{r+l}$ of $\mathfrak{t}$, and $x_{r+l+1}, \dots, x_N$ of $\mathfrak{n}^-$ we see that $U\mathfrak{g} = U\mathfrak{n}^- \otimes U\mathfrak{t} \otimes U\mathfrak{n}^+$ as a vector space. But $U\mathfrak{n}^+ \cdot v = \mathbb{C}v$ (as $\mathfrak{n}^+ \cdot v = 0$) and $U\mathfrak{t} \cdot v = \mathbb{C}v$, thus $U\mathfrak{g} = U\mathfrak{n}^-$. $\square$

Remark 7.35: If V is irreducible and finite-dimensional, then it is a highest weight module.

Proposition 7.36: Let V be a highest weight module for $\mathfrak{g}$ (no necessarily finite-dimensional), and let v_Λ be a highest weight vector with highest weight $\Lambda \in \mathfrak{t}^*$, then:

(i) $\mathfrak{t}$ acts diagonalizable on V, and $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$, where

$$D(\Lambda) = \{\Lambda - \sum k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0}\} = \{\mu \in \mathfrak{t}^* \mid \mu \leq \Lambda\}.$$

$D(\Lambda)$ is called the *descent* of Λ.

(ii) $V_\Lambda = \mathbb{C}v_\Lambda$, and all other weight spaces are finite-dimensional.

(iii) V is irreducible if and only if all singular vectors are in V_Λ.

(iv) Ω acts on V as $|\Lambda + \rho|^2 - |\rho|^2$.

(v) If v_λ is any singular vector in V, then $|\lambda + \rho| = |\Lambda + \rho|$

(vi) There exist only finitely many λ such that V_λ contains a singular vector.

(vii) V contains a unique maximal proper submodule I, I is graded by $\mathfrak{t}$ (i.e. $I = \oplus_{\lambda \in \mathfrak{t}^*}(I \cap V_\lambda)$), and I is the sum of all proper submodules of V.

Proof:

(i),(ii) As $V = U\mathfrak{n}^- \cdot v_\Lambda$, expressions of the form $e_{-\beta_1} e_{-\beta_2} \dots e_{-\beta_r} v_\Lambda$ span V, where $\beta_i \in R^+$ and $e_{-\beta_i} \in \mathfrak{g}_{-\beta_i}$. But the weight of such an expression is $\Lambda - \beta_1 - \beta_2 - \dots - \beta_r$ (*Exercise*: proof this, note $tev = [t, e]v + etv = (-\beta(t) + \Lambda(t))ev$). Whence (i) and (ii) hold as there are only finitely many $\beta \in R^+$ which sum up to a given weight $\lambda \in \mathbb{Z}_{\geq 0} R^+$.

(iii) If $v_\lambda \in V_\lambda$ is a singular vector, then $N = U\mathfrak{g} \cdot v_\lambda = U\mathfrak{n}^- \cdot v_\lambda$ is a submodule of V, whose weights, are in $D(\lambda)$, by (i). But $\lambda \neq \Lambda$ implies $D(\lambda) \subsetneq D(\Lambda)$, so N is a proper submodule as $v_\Lambda \notin N$, i.e. V is not irreducible.
Conversely, if $N \subsetneq V$ is a proper submodule, then, as $\mathfrak{t}N \subseteq N$, N is graded by $\mathfrak{t}$, and its weights are in $D(\Lambda)$. Let $\lambda = \Lambda - \sum k_i \alpha_i$ be a weight of N ($\alpha_i \in \Pi$) and $\sum k_i$ minimal. Then $\sum k_i > 0$ as otherwise $\Lambda = \lambda$ and $N = V$. Now, if $0 \neq v \in N_\lambda$, then v is singular as for $\alpha \in R^+$, $e_\alpha \cdot v \in N_{\lambda+\alpha}$, but $N_{\lambda+\alpha} = 0$ by minimality of $\sum k_i$.

(vii) Any proper submodule of V is $\mathfrak{t}$-graded and does not contain v_Λ. Therefore, the sum of all proper submodules still does not intersect V_Λ and is $\mathfrak{t}$-graded, so it is the maximal proper submodule.

(iv) We know from 7.25 that for any singular vector v_Λ we have

$$\Omega v_\Lambda = (|\Lambda + \rho|^2 - |\rho|^2) v_\Lambda.$$

Moreover, Ω is central, so $\Omega e_{-\beta_1} \dots e_{-\beta_r} v_\Lambda = e_{-\beta_1} \dots e_{-\beta_r} \Omega v_\Lambda$ and these elements span V. Therefore, we see that Ω acts by the same constant on all of V.

(v) Follows immediately from (iv) and 7.25 by applying Ω to a singular vector with weight λ.

(vi) If V_λ contains a singular vector, then $|\lambda+\rho|^2 = |\Lambda+\rho|^2$. This equation defines a sphere in $\mathbb{R}R$ (center ρ, radius $|\Lambda+\rho|$) — a compact set. On the other hand, $D(\Lambda)$ is discrete, and the intersection of a compact and a discrete set is finite. □

Definition 7.37: Let $\Lambda \in \mathfrak{t}^*$. A *Verma module* with highest weight Λ and highest weight vector v_Λ, $M(\Lambda)$, is a universal module with highest weight Λ, i.e. if V is any highest weight module with highest weight vector v (also of weight Λ), there exists a unique map

$$M(\Lambda) \to V,\ v_\Lambda \mapsto v.$$

Proposition 7.38: Let $\Lambda \in \mathfrak{t}^*$, then:

(i) There exists a unique Verma module $M(\Lambda)$.

(ii) There exists a *unique* irreducible highest weight module of weight Λ, we denote it by $L(\Lambda)$.

Proof:

(i) Uniqueness of $M(\Lambda)$ is clear by the universal property. For existence, define

$$M(\Lambda) = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\Lambda,$$

where $\mathfrak{b} = \mathfrak{n}^+ + \mathfrak{t}$ and $\mathbb{C}_\Lambda$ is the $\mathfrak{b}$-module on which $\mathfrak{n}^+ \cdot v = 0$ and $\mathfrak{t} \cdot v = \Lambda(t)v$, i.e.

$$M(\Lambda) = U\mathfrak{g}/J(\Lambda),$$

where $J(\Lambda)$ is the left ideal generated by $u - \Lambda(u)$ for all $u \in U\mathfrak{b}$. Here we extend Λ to the character $U\mathfrak{b} \to \mathbb{C}$. In other words, $M(\Lambda)$ is the module generated by $\mathfrak{g}$ acting on 1, with relations $\mathfrak{n}^+ \cdot 1 = 0, \mathfrak{t} \cdot 1 = \Lambda(t)1$, for all $t \in \mathfrak{t}$ and only the relations these imply. So if V is an arbitrary highest weight module with weight Λ, it is clear that $V = U\mathfrak{g}/J$, where J is some ideal containing $J(\Lambda)$, i.e. $M(\Lambda) \twoheadrightarrow V$.

(ii) From the proof of (i) follows in particular, that an irreducible highest-weight module must be of the form $M(\Lambda)/I(\Lambda)$ where $I(\Lambda)$ is a *maximal* proper submodule of $M(\Lambda)$. But we have just shown that there is an unique maximal proper submodule, so $L(\Lambda)$ is unique. □

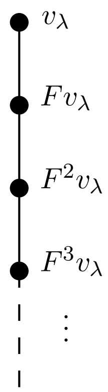
Proposition 7.39: Let $R^+ = \{\beta_1, \ldots, \beta_r\}$, then $e_{-\beta_1}^{k_1} \cdots e_{-\beta_r}^{k_r} v_\Lambda$ is a basis of $M(\Lambda)$.

Proof: The "hard" part of the PBW Theorem 7.30 implies this immediately. □

Corollary 7.40: Any irreducible finite-dimensional $\mathfrak{g}$-module is of the form $L(\Lambda)$ for some $\Lambda \in P^+$.

Proof: We know that it is of the form $L(\Lambda)$, some Λ, and we have seen that the highest weight must be in P^+ by results of the $\mathfrak{sl}_2$ theory. □

Example 7.41: Let $\mathfrak{g} = \mathfrak{sl}_2$. The Verma module $M(\Lambda)$ is an infinite string of the following shape:



Exercise 7.42 (Essential): Let first $\mathfrak{g} = \mathfrak{sl}_2$.

(i) Show that $M(\lambda) = L(\lambda)$ (i.e. $M(\lambda)$ is irreducible) if and only if $\lambda \notin \mathbb{Z}_{\geq 0}$.

(ii) Show that if $\lambda \in \mathbb{Z}_{\geq 0}$, then $M(\lambda)$ contains a unique proper submodule, the span of $F^{\lambda+1}v_\lambda$, $F^{\lambda+2}v_\lambda, \ldots$. This submodule is itself a Verma module.

(iii) Let now $\mathfrak{g}$ be an arbitrary simple Lie algebra, and $\Lambda(H_i) \in \mathbb{Z}_{\geq 0}$. Show that $F_i^{\Lambda(H_i)+1} \cdot v_\Lambda$ is a singular vector of $M(\Lambda)$ (NB: there will also be other singular vectors).

(iv) Very important: compute $\operatorname{ch} M(\Lambda)$.

Proposition 7.43: Let $\Lambda \in P^+$, then $L(\Lambda)$ is *integrable*, i.e. E_i and F_i act locally nilpotently (that is, all $v \in L(\Lambda)$ are contained in a finite-dimensional subspace on which E_i acts nilpotently, i.e. $E_i^n v = 0$ for some $n > 0$, and ditto for F_i).

Proof: If V is any highest-weight module, then E_i acts locally nilpotently (as $E_i V_\lambda \subseteq V_{\lambda+\alpha_i}$, but weights of V are in the cone $D(\Lambda) = \{\Lambda - \sum k_i\alpha_i \mid k_i \geq 0\}$). We must show that F_i acts locally nilpotently. We know that $F_i^{\Lambda(H_i)+1} \cdot v_\Lambda$ is a singular vector, by Exercise 7.42 (iii). But $L(\Lambda)$ is irreducible, so it has no singular vectors other than v_Λ, so $F_i^{\Lambda(H_i)+1} v_\Lambda = 0$ by the following exercise, which finishes the proof. □

Exercise 7.44:

(i) $a^k b = \sum_{i=0}^{k} \binom{k}{i} ((\operatorname{ad} a)^i b) a^{k-i}$

(ii) Using (i) and the Serre relations $(\operatorname{ad} e_\alpha)^4 e_\beta = 0$ (for all $\alpha, \beta \in R$), show $F_i^N e_{-\beta_1} \ldots e_{-\beta_r} v_\Lambda = 0$ for $N \gg 0$ by induction on r.

Note that we need the power 4 in the Serre relations in the worst case, for G_2 where we have a string $\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta$. Notice that this is true for a generalized Kac-Moody algebra as well, but then the 4 is replaced by the maximal $-a_{ij}+1$ in the Cartan matrix.

Corollary 7.45: We have $\dim L(\Lambda)_\mu = \dim L(\Lambda)_{w\mu}$ for all $w \in W$.

Proof: We have seen that E_i, F_i act locally nilpotently implies this statement is true for $w = s_{\alpha_i}$ — a simple reflection (see proof sketch 2 of 7.8). But W is generated by $s_{\alpha_1}, \ldots s_{\alpha_l}$, so this even holds for all $w \in W$. □

Theorem 7.46 (Cartan's theorem): If $\mathfrak{g}$ is finite-dimensional and $\Lambda \in P^+$, then $L(\Lambda)$ is finite-dimensional.

Proof: Let $\alpha \in R^+$. We know that e_α acts nilpotently on $L(\Lambda)$. We show first, that also $e_{-\alpha}$ does, to see that all of the root $\mathfrak{sl}_2$-copies act integrably. In fact, $e_{-\alpha}^n v_\Lambda = 0$ for $n = \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} = (\Lambda, \alpha^\vee) + 1$, as if not, we would have $L(\Lambda)_{\Lambda - n\alpha} \neq 0$ and hence by Corollary 7.45 that

$$\begin{aligned} s_\alpha(\Lambda - n\alpha) &= s_\alpha(\Lambda) + n\alpha \\ &= \Lambda - (\Lambda, \alpha^\vee)\alpha + ((\Lambda, \alpha^\vee) + 1)\alpha \\ &= \Lambda + \alpha > \Lambda \end{aligned}$$

is also a weight in $L(\Lambda)$, contradicting that Λ is the highest weight. Thus, by Exercise 7.44, we see that $e_{-\alpha}$ acts locally nilpotently on all of $L(\Lambda)$. Therefore,

$$U(\mathfrak{n}^-)v_\Lambda = \langle e_{-\beta_1}^{k_1} \dots e_{-\beta_r}^{k_r} v_\Lambda \rangle = L(\Lambda), \quad \text{for } R^+ = \{\beta_1, \dots \beta_r\},$$

is finite-dimensional. □

Now we can prove the complete reducibility stated in Theorem 7.14. In order to do that, we need the following lemmas:

Lemma 7.47: For the reflection $s_i = s_{\alpha_i}$ of the i-th simple root α_i the condition $s_i(R^+ \setminus \{\alpha_i\}) = R^+ \setminus \{\alpha_i\}$ holds.

Proof: Let $\alpha = \sum_j k_j \alpha_j \in R^+$, then all $k_j \geq 0$. Now

$$s_i \alpha = \sum_{j \neq i} k_j \alpha_j - \left(\sum_{j \neq i} (\alpha_j, \alpha_i^\vee) k_j + k_i \right) \alpha_i,$$

but $\alpha \neq \alpha_i$, so some $k_j > 0, j \neq i$. Thus, the coefficient of α_j in $s_i\alpha$ is still positive (as it is the same coefficient k_j). But $R = R^+ \coprod (-R^+)$, i.e. the disjoint union of roots with all coefficients ≥ 0 and roots where all coefficients are ≤ 0. This implies $s_i \alpha \in R^+$, and $s_i \alpha \neq \alpha_i$. □

Recall $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.

Lemma 7.48: We have $\rho(H_i) = 1$, for all i, i.e. $\rho = \omega_1 + \ldots + \omega_l \in P^+$.

Proof: Observe

$$\begin{aligned} s_i \rho &= s_i \left(\frac{1}{2}\alpha_i + \frac{1}{2} \sum_{\substack{\alpha \neq \alpha_i \\ \alpha \in R^+}} \alpha \right) \\ &= -\frac{1}{2}\alpha_i + \frac{1}{2} \sum_{\substack{\alpha \neq \alpha_i \\ \alpha \in R^+}} \alpha = \rho - \alpha_i. \end{aligned}$$

But, $s_i \rho = \rho - (\rho, \alpha_i^\vee)\alpha_i$, so $(\rho, \alpha_i^\vee) = 1$ for all i. □

Lemma 7.49 (Key lemma): Let $\Lambda \in P^+$, and $\mu \leq \Lambda$ such that $\mu + \rho \in P^+$. Then

$$|\Lambda + \rho| = |\mu + \rho| \implies \Lambda = \mu.$$

Proof: Denote $\Lambda - \mu = \sum k_i \alpha_i$, then for all i, $k_i \geq 0$, and we compute

$$\begin{aligned} 0 &= (\Lambda + \rho, \Lambda + \rho) - (\mu + \rho, \mu + \rho) = (\Lambda + \rho - (\mu + \rho), \Lambda + \rho + \mu + \rho) \\ &= (\Lambda - \mu, \Lambda + \rho + \mu + \rho) = \sum k_i (\alpha_i, \underbrace{\Lambda + \rho + \mu + \rho}_{\in P^+}), \end{aligned}$$

as $\Lambda, \rho, \mu + \rho \in P^+$. But $(\alpha_i, (\Lambda + \mu) + 2\rho) \geq 1$. This implies $k_i = 0$ for all i. □

Theorem 7.50 (Weyl complete reducibility, cf. 7.14): Let $\operatorname{char} k = 0$, $k = \overline{k}$ and $\mathfrak{g}$ be a semisimple Lie algebra over k, then every finite-dimensional $\mathfrak{g}$-module V is a direct sum of irreducibles.

Proof: Recall that V is completely reducible as an $(\mathfrak{sl}_2)_\alpha$-module. Write $V = \oplus_{\lambda \in P} V_\lambda$. Consider $V^{\mathfrak{n}^+} = \{v \in V \mid \mathfrak{n}^+ v = 0\}$. By Engel's theorem, $V^{\mathfrak{n}^+} \neq 0$, and $[\mathfrak{t}, \mathfrak{n}^+] \subseteq \mathfrak{n}^+$. Hence $\mathfrak{t}$ acts on $V^{\mathfrak{n}^+}$, and so $V^{\mathfrak{n}^+} = \bigoplus_{\mu \in P} V_\mu^{\mathfrak{n}^+}$, where $V_\mu^{\mathfrak{n}^+} = \{x \in V \mid \mathfrak{n}^+ x = 0 \text{ and } tx = \mu(t)x\}$. Therefore, $V^{\mathfrak{n}^+}$ consists of singular vectors.

We claim that for every $0 \neq v_\mu \in V_\mu^{\mathfrak{n}^+}$, the module $L = U\mathfrak{g} \cdot v_\mu$ is irreducible. To prove this, note that L is a highest weight module with highest weight μ, so we must only show that it has no other singular vectors. If λ is the weight of a singular vector in L, then $\lambda \leq \mu$, but also $|\lambda + \rho| = |\mu + \rho|$ by considering the action of the Casimir. Since V, and therefore L, is finite-dimensional, we must have $\lambda, \mu \in P^+$ (by 7.14, and $\lambda(h_i) = (\lambda, \alpha_i^\vee)$). So by the key lemma, $\lambda = \mu$.

It follows that $V' = U\mathfrak{g} \cdot V^{\mathfrak{n}^+}$ is completely reducible (as if $\{v_1, \ldots, v_r\}$ is a basis of weight vectors for $V^{\mathfrak{n}^+}$ with weights $\lambda_1, \ldots, \lambda_r$, then $V' = L(\lambda_1) \oplus \ldots \oplus L(\lambda_r)$). So to finish, we must show that $N = V/V' = 0$.

If $N \neq 0$, then $N^{\mathfrak{n}^+} \neq 0$. Let $v_\lambda \in N_\lambda^{\mathfrak{n}^+}$ be a singular vector, as N is finite-dimensional, $\lambda \in P^+$. Lift v_λ to $\overline{v_\lambda} \in V_\lambda$, then $E_i \overline{v_\lambda} \in V_{\lambda + \alpha_i}$ and there exist some i s.t. $E_i \overline{v_\lambda} \neq 0$ as otherwise $\overline{v_\lambda}$ is a singular vector and $U\mathfrak{g}\overline{v_\lambda}$ is contained in V', contradicting our choice of $v_\lambda \neq 0$. But then, as $E_i \overline{v}_\lambda \in V'$,

$$\Omega E_i \overline{v_\lambda} \overset{7.25}{=} E_i \Omega \overline{v_\lambda} = \left(\left|\lambda' + \rho\right|^2 - |\rho|^2\right) E_i \overline{v_\lambda}, \quad \lambda' \in \{\lambda_1, \ldots, \lambda_r\},$$

but on the other hand, as v_λ is a singular vector in N, $\Omega \overline{v_\lambda} = |\lambda + \rho|^2 - |\rho|^2 \, \overline{v_\lambda}$, so $|\lambda' + \rho| = |\lambda + \rho|$ (by 7.36). Moreover, $\lambda + \alpha_i$ is a weight in $L(\lambda')$, so $\lambda + \alpha_i = \lambda' - \sum k_j \alpha_j$, for some k_j, i.e. $\lambda = \lambda' + \sum k_i' \alpha_i$ with not all k_i' zero contradicting the key lemma, so $V = V'$. □

7.3 The Weyl character formula

Lemma 7.51: Let $\Lambda \in \mathfrak{t}^*$ and $M(\Lambda)$ the Verma module of Λ. Then

$$\operatorname{ch} V(\Lambda) = \frac{e^\Lambda}{\prod_{\alpha \in R^+} (1 - e^{-\alpha})}. \tag{10}$$

Proof: Let $R^+ = \{\beta_1, \ldots, \beta_r\}$. The PBW Theorem 7.30 gives the basis $\{e_{-\beta_1}^{k_1} \cdots e_{-\beta_r}^{k_r} v_\Lambda \mid k_i \in \mathbb{Z}_{\geq 0}\}$ for the Verma module $M(\Lambda)$ and the weight of such an element is $\Lambda - \sum_{i=1}^r k_i \beta_i$. So the dimension of a weight space $M(\Lambda)_{\Lambda - \beta}$ is the number of ways of writing β as $\sum k_i \beta_i$. But this is the coefficient of $e^{-\beta}$ in $\prod_{\alpha \in R^+} (1 - e^\alpha)^{-1}$. □

Let us write $\Delta = \prod_{\alpha\in R^+}(1-e^{-\alpha})$. We have just shown that $\operatorname{ch} M(\Lambda) = e^{\Lambda}/\Delta$.

Lemma 7.52: For all $w \in W$, $w(e^{\rho}\Delta) = \det w \cdot e^{\rho}\Delta$. Here, $\det\colon W \to \mathbb{Z}/2$ is the determinant of w acting on $\mathfrak{t}^*$ ($= \pm 1$).

Proof: Since W is generated by simple reflections, it is enough to show that $s_i(e^{\rho}\Delta) = -e^{\rho}\Delta$. But

$$\begin{aligned} s_i(e^{\rho}\Delta) &= s_i\left(e^{\rho}(1-e^{-\alpha_i})\prod_{\substack{\alpha\neq\alpha_i\\ \alpha\in R^+}}(1-e^{-\alpha})\right)\\ &= e^{\rho-\alpha_i}(1-e^{+\alpha_i})\prod_{\substack{\alpha\neq\alpha_i\\ \alpha\in R^+}}(1-e^{-\alpha}) = -e^{\rho}\Delta, \end{aligned}$$

as $s_i\rho = \rho - \alpha_i$, and $s_i(R^+\setminus\{\alpha_i\}) = R^+\setminus\{\alpha_i\}$. □

Lemma 7.53: For any highest weight module $V(\Lambda)$ with highest weight Λ

(i) there exist coefficients $a_\lambda \geq 0$, $\lambda \leq \Lambda$, such that

$$\operatorname{ch} V(\Lambda) = \sum_{\substack{\lambda\leq\Lambda\\ |\lambda+\rho|=|\Lambda+\rho|}} a_\lambda \operatorname{ch} L(\lambda), \quad \text{with } a_\Lambda = 1; \tag{11}$$

(ii) there exist coefficients $b_\lambda \in \mathbb{Z}$ with $b_\Lambda = 1$ such that

$$\operatorname{ch} L(\Lambda) = \sum_{\substack{\lambda\leq\Lambda\\ |\lambda+\rho|=|\Lambda+\rho|}} b_\lambda \operatorname{ch} M(\lambda). \tag{12}$$

Proof: (i)⇒(ii): We write $B(\Lambda) = \{\lambda \leq \Lambda \mid |\lambda+\rho| = |\Lambda+\rho|\}$. Recall that $B(\Lambda)$ is a finite set (for $\Lambda \in \mathbb{R}R$). We have a total order on $B(\Lambda) = \{\lambda_1,\dots,\lambda_n\}$ so that if $\lambda_i \leq \lambda_j$, then $i \leq j$. Then (i) is a system of equations relating $\operatorname{ch} M(\lambda)$ and $\operatorname{ch} L(\lambda)$, which is upper-triangular with ones on the diagonal and therefore invertible. Inverting this system gives (ii).
(i): Recall that the weight spaces of a highest weight module are finite-dimensional. We induct on $\sum_{\mu\in B(\Lambda)} \dim V(\Lambda)_\mu$. Note that if $V(\Lambda)$ is irreducible, (i) is true with $a_\Lambda = 1, a_\lambda = 0$, if $\lambda \neq \Lambda$. Otherwise, there exists a root $\mu \in B(\Lambda)$ with a singular vector $v_\mu \in V(\Lambda)_\mu$. Choose μ so that the height ($\sum k_i$) of $\Lambda - \mu = \sum k_i\alpha_i$ is maximal for all singular vectors. Then $L(\mu) := U\mathfrak{g}\cdot v_\mu \subseteq V(\Lambda)$ has no singular vectors, and is therefore irreducible. Set $\overline{V(\Lambda)} = V(\Lambda)/L(\mu)$ (i.e there exists an exact sequence $0 \to L(\mu) \to V(\Lambda) \to \overline{V(\Lambda)} \to 0$), then we see that $\overline{V(\Lambda)}$ is a highest-weight module with a smaller value of $\sum_{\mu\in B(\Lambda)} \dim \overline{V(\Lambda)}_\mu$, and $\operatorname{ch} V(\Lambda) = \operatorname{ch}\overline{V(\Lambda)} + \operatorname{ch} L(\mu)$. So we are done by induction. □

We will now compute $\operatorname{ch} L(\Lambda)$ for $\Lambda \in P^+$. We know that

$$\operatorname{ch} L(\Lambda) = \sum_{\lambda\in B(\Lambda)} b_\lambda \operatorname{ch} M(\lambda) = \sum_{\lambda\in B(\Lambda)} b_\lambda \frac{e^{\lambda}}{\Delta}.$$

Further, we have seen before that $w(\operatorname{ch} L(\Lambda)) = \operatorname{ch} L(\Lambda)$ for all $w \in W$, and $w(\Delta e^\rho) = \det w \cdot \Delta e^\rho$. Therefore,

$$e^\rho \Delta \operatorname{ch} L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda+\rho}$$

is W-anti-invariant. So $w\left(\sum_{\lambda\in B(\Lambda)} b_\lambda e^{\lambda+\rho}\right) = \det w. \left(\sum_{\lambda\in B(\Lambda)} b_\lambda e^{\lambda+\rho}\right)$. Let us rewrite this as

$$\sum_{\lambda\in B(\Lambda)} b_\lambda e^{\lambda+\rho} = \sum_{\lambda_1,\dots,\lambda_s} b_\lambda \sum_{w\in W} \det w \cdot e^{w(\lambda+\rho)},$$

where $\lambda_1, \dots, \lambda_s$ is a representative system for the orbits of W acting on $B(\Lambda + \rho)$. Now, if $\lambda \in \mathbb{R}R$ (which is true, since $\Lambda \in P^+$), then $W(\lambda + \rho)$ intersects $\{x \in \mathbb{R}R \mid (x, \alpha_i^\vee, \forall i) \geq 0\}$ in exactly one point (this set is a fundamental domain for the W-action on $\mathbb{R}R$). (Note that for a given $\lambda \in \mathbb{R}R$, $W(\lambda+\rho)$ defines a positive root system and W acts simply transitively on those roots). Therefore, we can take a representative system for the orbits only containing dominant weights. Note that one of these dominant weights is Λ and the other orbits are given by W acting on $\{\lambda \in B(\Lambda) \mid \lambda \neq \Lambda, \lambda \in P^+\}$. But the key lemma 7.49 implies that this set is empty, so the only coefficient is $b_\Lambda = 1$. This proves the following theorem:

Theorem 7.54 (Weyl Character Formula): For all $\Lambda \in P^+$

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w\in W} \det w \cdot e^{w(\Lambda+\rho)}}{e^\rho \prod_{\alpha\in R^+}(1 - e^{-\alpha})} \tag{13}$$

$$= \sum_{w\in W} \det w \cdot \operatorname{ch} M(w(\Lambda+\rho) - \rho). \tag{14}$$

Example 7.55: Let $\mathfrak{g} = \mathfrak{sl}_2$, and write $z = e^{\alpha/2}$, then $\mathbb{C}[P] = \mathbb{C}[z, z^{-1}]$ and $e^\rho = z$, and we have

$$\operatorname{ch} L(m\frac{\alpha}{2}) = \frac{z^{m+1} - z^{-(m+1)}}{z - z^{-1}}$$

as we saw earlier in this course.

Corollary 7.56 (Weyl denominator identity): As $L(0) = \mathbb{C}$, we have $\operatorname{ch} L(0) = 1$, so

$$e^\rho \prod_{\alpha\in R^+}(1 - e^{-\alpha}) = \sum_{w\in W} \det w \cdot e^{w\rho}.$$

Exercise 7.57: Let $\mathfrak{g} = \mathfrak{sl}_n$. Show that the Weyl denominator identity is equivalent to the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \dots & z_n^{n-1} \end{pmatrix} = \prod_{i<j}(z_i - z_j),$$

where we write $z_i = e^{e_i}$.

Corollary 7.58 (Weyl dimension formula): For all $\Lambda \in P^+$ we have

$$\dim L(\Lambda) = \prod_{\alpha\in R^+} \frac{(\alpha, \Lambda+\rho)}{(\alpha, \rho)}. \tag{15}$$

Example 7.59: $\mathfrak{g} = \mathfrak{sl}_3$ (root system is of type A_2), $R^+ = \{\alpha, \beta, \alpha+\beta\}$ with $\rho = \alpha+\beta = \omega_1+\omega_2$. Let $\Lambda = m_1\omega_1 + m_2\omega_2$, then

	α	β	$\alpha+\beta$
$(\cdot, \Lambda+\rho)$	m_1+1	m_2+1	m_1+m_2+2
$(\cdot, \rho)$	1	1	2

Therefore, $\dim L(\Lambda) = \frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2)$.

Exercise 7.60: Compute the dimensions of all the finite-dimensional irreducible representations of B_2 and G_2.

Remark 7.61: Let $w \in W$ be written as $w = s_{i_1}s_{i_2}\dots s_{i_r}$ where s_{i_k} are simple reflections. Then $\det w = (-1)^r$. The minimal r such that w can be written in this form is called the *length* of w, denoted $l(w)$. The *Monoid Lemma* asserts that you can get from one minimal-length expression for w to another by repeatedly applying the braid relations.

Exercise 7.62: $l(w) = \#\{-R^+ \cap w^{-1}R^+\} = l(w^{-1})$.

Proof: (Weyl dimension formula). We still have to prove the Weyl dimension formula 7.58. We know $\operatorname{ch} L(\Lambda) = \sum \dim L(\Lambda)_\lambda e^\lambda \in \mathbb{C}[P]$. We would like to set $e^\lambda \mapsto 1$, but then the denominator in the Weyl character formula would become 0. Instead, consider the homomorphism

$$F_\mu \colon \mathbb{C}[P] \to \mathbb{C}(q), e^\lambda \mapsto q^{-(\lambda,\mu)}.$$

For example, $F_0(e^\lambda) = 1$, so $F_0(\operatorname{ch} L(\lambda)) = \dim L(\lambda)$. Now apply F_μ to the Weyl dominator identity. Then

$$q^{-(\rho,\mu)} \prod_{\alpha\in R^+} (1 - q^{(\alpha,\mu)}) = \sum_{w\in W} \det w q^{-(w\rho,\mu)} = \sum_{w\in W} \det w q^{-(\rho,w\mu)}$$

as $\det w = \det w^{-1}$ and $(x, wy) = (w^{-1}x, y)$ (i.e. the Weyl group is a subgroup of the orthogonal group of the inner product). We now apply F_μ to the Weyl character formula:

$$F_\mu(\operatorname{ch} L(\Lambda)) = \frac{\sum_{w\in W} \det w q^{-(w(\Lambda+\rho),\mu)}}{q^{-(\rho,\mu)} \prod_{\alpha\in R^+}(1 - q^{(\alpha,\mu)})}$$

if $(\alpha, \mu) \neq 0$ for all $\alpha \in R^+$.

Now, take $\mu = \rho$ (recall that $(\rho, \alpha_i) = 1 > 0$ for all simple roots α_i, so $(\rho, \alpha) > 0$ for all $\alpha \in R^+$), so

$$F_\rho(\operatorname{ch} L(\Lambda)) = \sum \dim L(\lambda)_\lambda q^{-(\lambda,\rho)} = \frac{q^{-(\rho,\Lambda+\rho)} \prod_{\alpha\in R^+}(1 - q^{(\alpha,\Lambda+\rho)})}{q^{-(\rho,\rho)} \prod_{\alpha\in R^+}(1 - q^{(\alpha,\rho)})}$$

where we used our expression for the Weyl denominator identity and applied it to the numerator. From this we can conclude the Weyl dimension formula

$$\dim L(\Lambda) = \prod_{\alpha\in R^+} \frac{(\Lambda+\rho, \alpha)}{(\rho, \alpha)}$$

by setting $q = 1$ and applying L'Hôpital's rule. $\square$

Remark 7.63: We can now algorithmically answer all questions about finite dimensional representations of semisimple Lie algebras by knowing the highest weight. For example, let us decompose $L(\lambda) \otimes L(\mu) = \sum m^{\nu}_{\lambda\mu} L(\nu)$ (by complete reducibility). To compute the *Littlewood-Richardson coefficients* $m^{\nu}_{\lambda\mu}$ (recall that we had the Clebsch-Gordan rule for them in $\mathfrak{sl}_2$) define

$$\begin{aligned} \bar{\ }\colon \mathbb{Z}[P] &\to \mathbb{Z}[P],\ e^{\lambda} \mapsto e^{-\lambda}, \\ CT\colon \mathbb{Z}[P] &\to \mathbb{Z},\ e^{\lambda} \mapsto \begin{cases} 0 & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0 \end{cases}, \text{ and} \\ (\cdot,\cdot)\colon \mathbb{Z}[P] \times \mathbb{Z}[P] &\to \mathbb{Z},\ (f,g) = \frac{1}{|W|} CT(f\overline{g}\Delta\overline{\Delta}), \end{aligned}$$

where $\Delta = \prod_{\alpha \in R^+}(1 - e^{-\alpha})$.
Claim: if we let $\chi_\lambda = \operatorname{ch} L(\lambda)$, then $(\chi_\lambda, \chi_\mu) = \delta_{\mu\nu}$, and thus $m^{\nu}_{\lambda\mu} = (\chi_\lambda\chi_\mu, \chi_\nu)$.

Proof:

$$(\chi_\lambda, \chi_\mu) = \frac{1}{|W|} CT\Big(\sum_{x,w \in W} e^{w(\lambda+\rho)-\rho}\overline{e^{x(\mu+\rho)-\rho}} \det(wx)\Big)$$

by Weyl. But $CT(e^{w(\lambda+\rho)-x(\mu+\rho)}) = \delta_{wx}\delta_{\mu\lambda}$ as for $\lambda, \mu \in R^+$ we have $w(\lambda+\rho) = \mu + \rho$ if and only if $w = 1, \mu = \lambda$ (as the dominant weights are the lattice points in a fundamental domain of the W-action), so $x^{-1}w(\lambda+\rho) = \mu + \rho$ precisely if $x = w, \lambda = \mu$. □

7.4 Principal $\mathfrak{sl}_2$

Define $\rho^\vee \in \mathfrak{t}^*$ by $(\rho^\vee, \alpha_i) = 1$ for all $\alpha_i \in \Pi$ (recall that $(\rho, \alpha_i^\vee) = 1$, so $\rho^\vee$ can be seen as ρ for $R^\vee$).

Exercise 7.64: Show $\rho^\vee = \frac{1}{2}\sum_{\alpha \in R^+} \alpha^\vee$. In particular, if R is simply laced, $\rho = \rho^\vee$. This implies $(\rho^\vee, \alpha) = \operatorname{ht}(\alpha) = \sum k_i$ if we write $\alpha = \sum k_i\alpha_i$.

Exercise 7.65:

$$F_{\rho^\vee}(\operatorname{ch} L(\Lambda)) = q^{-(\Lambda,\rho^\vee)} \prod_{\alpha \in R^+} \frac{(1 - q^{(\Lambda+\rho,\alpha^\vee)})}{(1 - q^{(\rho,\alpha^\vee)})}.$$

Hint: Apply $F_{\rho^\vee}$ to the Weyl denominator identity of the irreducible representations of the Lie algebra with root system $R^\vee$. Note that $\frac{(\lambda+\rho,\alpha^\vee)}{(\rho,\alpha^\vee)} = \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$ as $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, so we recover

Definition 7.66: We call $F_{\rho^\vee}(\operatorname{ch} L(\Lambda)) =: \dim_q L(\Lambda)$ the *q-dimension* of $L(\lambda)$.

Proposition 7.67: The q-dimension $\dim_q L(\Lambda)$ is a *unimodal polynomial*, i.e. it lives in $\mathbb{N}[q^2, q^{-2}]^{\mathbb{Z}/2}$ or $q\mathbb{N}[q^2, q^{-2}]^{\mathbb{Z}/2}$ (depending on its degree), and the coefficients decrease as the absolute value of the degree increases.

Proof: This follows if we show that $\dim_q L(\Lambda)$ is the character of an $\mathfrak{sl}_2$-module in which the length of all "strings" have the same parity. Let $H = 2\nu^{-1}(\rho^\vee) \in \mathfrak{t} \subseteq \mathfrak{g}$, and set $E = \sum E_i$. Check that $[H, E] = 2E$ (Exercise). Write $H = \sum c_i H_i$ for some $c_i \in \mathbb{C}$ ($H_1, \ldots, H_l$ is a basis of $\mathfrak{t}$), and set $F = \sum c_i F_i$. It is left as an exercise to show:

(i) Show that E, F, H generate a subalgebra isomorphic to $\mathfrak{sl}_2$, the so-called *principal* $\mathfrak{sl}_2$.

(ii) Show that if $\Lambda - \gamma$ is a weight of $L(\Lambda)$, then $(\Lambda - \gamma, 2\rho^\vee) \equiv (\Lambda, 2\rho^\vee) \pmod 2$.

This implies the proposition. □

Exercise 7.68: Write $[n] = \frac{q^n-1}{q-1}$. Show that the following polynomials are unimodal:

(i) $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$, where $[n]! = [n][n-1]\dots[1]$,

(ii) $(1+q)(1+q^2)\dots(1+q^n)$.

Hint: For (i), apply the above arguments to $\mathfrak{g} = \mathfrak{sl}_n$ and $V = S^k\mathbb{C}^n$ or $\Lambda^k\mathbb{C}^{n+k}$. For (ii), apply this to the spin representation of $B_n = \mathfrak{so}_{2n+1}$ (which we will define in 8.16).

Remark 7.69: An isomorphism $V \cong V^*$ implies a bilinear form $(\cdot,\cdot)$, but is this form in $\Lambda^2 V$ or $S^2 V$? Consider for example $\mathfrak{sl}_2$, it can be shown that the bilinear form induced by $L(n)$ is alternating precisely if n is odd, and symmetry if n is even. Notice that $L(\lambda) \cong L(\lambda)^*$ if and only if the lowest weight of $L(\lambda)$ is $-\lambda$ (for example, the $\mathbb{C}^2$ representations are always self-dual, and $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^3 + \mathbb{C}$). Now, the question whether the bilinear form induced by $L(\lambda) \otimes L(\lambda)^*$ is alternating or symmetric can be answered by checking this for the restriction to the principal $\mathfrak{sl}_2$. This is equivalent to $(\lambda, 2\rho^\vee)$ having odd or even parity.

Exercise 7.70:

(i) Compute $\dim_q L(\theta)$, where $L(\theta)$ is the adjoint representation, for A_2, B_2, and G_2. Then do this for all the classical groups.

(ii) You will notice that $L(\theta)|_{\text{principal } \mathfrak{sl}_2} = L(2e_1) + \dots + L(2e_l)$ where $l = \operatorname{rank}\mathfrak{g} = \dim\mathfrak{t}$, for some $e_1, \dots, e_l \in \mathbb{N}$ with $e_1 = 1$. The e_i are called the *exponents of the Weyl group.* Note that the order of the Weyl group is $|W| = (e_1+1)\dots(e_l+1)$. If you are in the mood, compute $|W|$ for E_8.

8 Crystals

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the simple roots, and P the weight lattice.

Definition 8.1: A *crystal* is a set B, $0 \notin B$, together with functions $\mathrm{wt}\colon B \to P$, $\tilde{e}_i\colon B \to B \sqcup \{0\}$, $\tilde{f}_i\colon B \to B \sqcup \{0\}$ such that

(i) If $\tilde{e}_i b \neq 0$, then $\mathrm{wt}\,\tilde{e}_i(b) = \mathrm{wt}\,b + \alpha_i$, and if $\tilde{f}_i b \neq 0$, then $\mathrm{wt}\,\tilde{f}_i(b) = \mathrm{wt}\,b - \alpha_i$.

(ii) For b and $b' \in B$, $\tilde{e}_i b = b'$ if and only if $b = \tilde{f}_i b'$.

(iii) $\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}\,b, \alpha_i^\vee \rangle$, for all $\alpha_i \in \Pi$, where

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\},$$
$$\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}.$$

We can draw B as a graph: The vertices are $b \in B$, and the edges are $b \xrightarrow[i]{} b'$ if $\tilde{e}_i b' = b$. We say that this edge is *coloured* by i. We call such a graph a *crystal graph.*

Example 8.2: Consider $\mathfrak{sl}_2$, then the string

$$n \longrightarrow n-2 \longrightarrow n-4 \longrightarrow \ldots \longrightarrow -n$$

is a crystal, where the weight of vertex i is $i\frac{\alpha}{2}$. Notice, for the crystal of the highest-weight representation $L(n) = L(nw_1)$, we have that if b is of weight $n-2k$, then $\varepsilon(b) = k$, and $\varphi(b) = n-k$ and the sum $\varepsilon_i(b) + \varphi_i(b)$ is the length of the string

$$\overbrace{n \longrightarrow n-2 \longrightarrow n-4 \longrightarrow \ldots \longrightarrow}^{\varepsilon} n-2k \overbrace{\longrightarrow \ldots \longrightarrow -n}^{\varphi}.$$

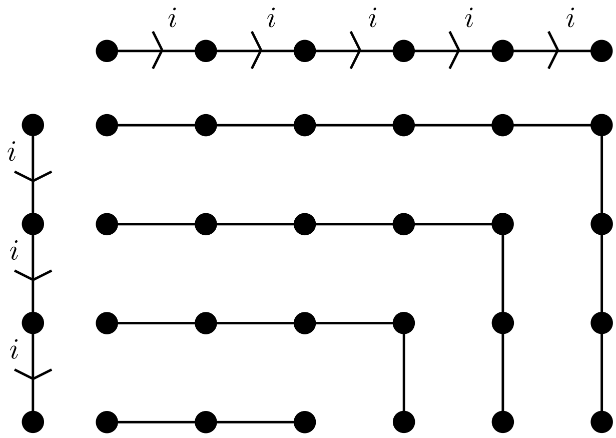
Define $B_\mu = \{b \in B \mid \operatorname{wt} b = \mu\}$.

If B_1 and B_2 are crystals, can define the tensor product $B_1 \otimes B_2 = B_1 \times B_2$ as a set, with $\operatorname{wt}(b_1 \otimes b_2) = \operatorname{wt} b_1 + \operatorname{wt} b_2$, and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes (\tilde{e}_i) b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \quad \text{whence}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes (\tilde{f}_i) b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

That is, in each colour i we have a graph of the form



the same form as we have seen for $\mathfrak{sl}_2$ before.

Exercise 8.3:

(i) Check that $B_1 \otimes B_2$ defines a crystal.

(ii) $B_1 \otimes (B_2 \otimes B_3) \cong (B_1 \otimes B_2) \otimes B_3$, $b_1 \otimes (b_2 \otimes b_3) \mapsto (b_1 \otimes b_2) \otimes b_3$. It suffices to prove this for $\mathfrak{sl}_2$. Note that it is not true in general that $B_1 \otimes B_2 \ncong B_2 \otimes B_1$.

Definition 8.4: $B^\vee$ is the crystal obtained from B by reversing the arrows. That is, $B^\vee = \{b^\vee \mid b \in B\}$, $\operatorname{wt} b^\vee = -\operatorname{wt} b$, $\varepsilon_i(b^\vee) = \varphi_i(b)$ (and vice versa), and $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ (and vice versa). In pictures:

$$\left(\bullet \underset{1}{\longrightarrow} \bullet \underset{2}{\longrightarrow} \bullet\right)^\vee = \left(\bullet \underset{2}{\longrightarrow} \bullet \underset{1}{\longrightarrow} \bullet\right).$$

Remark 8.5: If B corresponds to a basis of a representation V, then $B^\vee$ corresponds to a basis of the dual V^* as $B \to B^\vee$ comes from the Lie algebra anti-automorphism $e_i \mapsto f_i$, $f_i \mapsto e_i$, and $h_i \mapsto -h_i$. Notice that if $L(\lambda)$ is a representation with highest weight λ, then $L(\lambda)^*$ has lowest weight $-\lambda$.

Exercise 8.6: Show that $(B_1 \otimes B_2)^\vee = B_2^\vee \otimes B_1^\vee$.

Theorem 8.7 (Kashiwara): Let $L(\lambda)$ be the irreducible highest-weight representation with highest weight $\lambda \in P^+$, then:

(i) There exists a crystal $B(\lambda)$ whose elements are in 1-1 correspondence with a basis of $L(\lambda)$ (i.e. $B(\lambda)_\mu$ parametrizes a basis of $L(\lambda)_\mu$), so

$$\operatorname{ch} L(\lambda) = \sum_{b \in B(\lambda)} e^{wt(b)}. \tag{16}$$

(ii) For each simple root α_i (i.e. a simple $(\mathfrak{sl}_2)_i \subseteq \mathfrak{g}$), the decomposition of $L(\lambda)$ as an $(\mathfrak{sl}_2)_i$-module is precisely given by the i-coloured strings in $B(\lambda)$. (In particular, as an uncoloured graph, $B(\lambda)$ is connected, since it is spanned by elements of the form $\tilde{f}_1 \dots \tilde{f}_l \cdot v_\lambda$.)

(iii) The crystal $B(\lambda) \otimes B(\mu)$ is precisely the crystal for $L(\lambda) \otimes L(\mu)$, i.e. $B(\lambda) \otimes B(\mu)$ decomposes into connected components exactly in the way $L(\lambda) \otimes L(\mu)$ decomposes into irreducible representations.

Example 8.8: Let $\mathfrak{g} = \mathfrak{sl}_3$, $V = \mathbb{C}^3 = L(\omega_1)$, then the weight spaces are 1-dimensional, so we have no choice but to define the crystal as

$$w_1 \underset{1}{\rightarrow} w_1 - \alpha_1 \underset{2}{\rightarrow} w_1 - \alpha_1 - \alpha_2.$$

Let us compute $V \otimes V$ and $V \otimes V^*$:

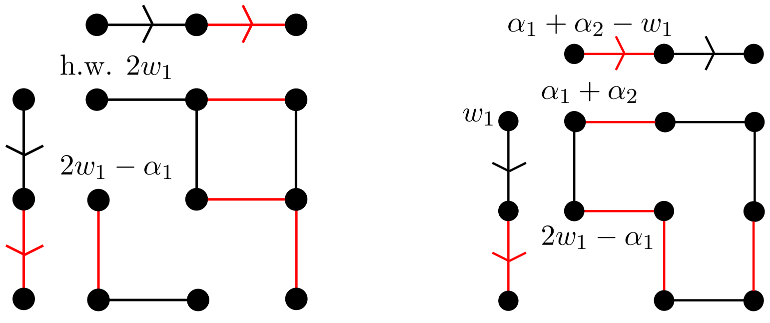


Figure 4: Crystals for $V \otimes V$ and $V \otimes V$, $V = \mathbb{C}^3$ as $\mathfrak{sl}_3$-module

Here, we chose black as colour 1, and red as colour 2 in the graphic. This implies,

$$V \otimes V = \mathbb{C}^3 \otimes \mathbb{C}^3 = S^2\mathbb{C}^3 + \Lambda^2\mathbb{C} = S^2\mathbb{C}^3 + (\mathbb{C}^3)^* \quad \text{as } 2w_1 - \alpha_1 = \alpha_1 + \alpha_2 - w_1, \text{ and}$$
$$V \otimes V^* = \operatorname{End} V = \mathbb{C} + \mathfrak{sl}_3 \text{ (as } \alpha_1 + \alpha_2 = \theta).$$

Remark 8.9: There are three proof approaches to the Kashiwara's theorem. The first one is due to Kashiwara, and is in the lecturer's opinion the most instructive.

Note that while the crystals give the decomposition of the representation into irreducibles, they do not correspond directly to a basis. That is, there is no $\mathfrak{sl}_2$-invariant basis that we could use here. Kashiwara's proof of the theorem uses the *quantum group* $U_q\mathfrak{sl}_2$, which is an algebra over $\mathbb{C}[q, q^{-1}]$ and a deformation of the universal enveloping algebra $U\mathfrak{sl}_2$. The two algebras $U\mathfrak{sl}_2$ and $U_q\mathfrak{sl}_2$ have the same representations, but over $\mathbb{C}[q, q^{-1}]$ there is a very nice basis which satisfies $e_i b = \tilde{e}_i b + q \cdot$ ("some mess"). Therefore, setting $q = 0$ ("freezing") will give the crystal.

A second proof approach is due to Lusztig. We will later look at the third proof using Littlemann paths, which give a purely combinatorial way of proving this theorem (which, on the face of it, is a purely combinatorial statement).

Definition 8.10: A crystal is called *integrable* if it is a crystal of an integrable highest-weight module with highest weight in P^+.

For two integrable crystals B_1, B_2, we do in fact have $B_1 \otimes B_2 = B_2 \otimes B_1$ (in general, this is false).

There is a combinatorial condition on crystals which implies that a crystal is integrable (due to Stembridge); it is a degeneration of the Serre relations.

8.1 Semi-standard Young tableaux

Consider $\mathfrak{sl}_n$:

$$\underset{e_1=w_1}{\bullet} \xrightarrow[1]{} \underset{w_1-\alpha_1}{\bullet} \xrightarrow[2]{} \underset{w_1-\alpha_1-\alpha_2}{\bullet} \xrightarrow[3]{} \ldots \xrightarrow[n-1]{} \bullet \xrightarrow[n-1]{}$$

is the crystal of the standard representation $L(w_1) = \mathbb{C}^n$. From this, we can construct the crystals for all $\mathfrak{sl}_n$-representations:

Let $\lambda \in P^+$, $\lambda = k_1 w_1 + \ldots + k_{n-1} w_{n-1}$, then $L(\lambda)$ is a summand of $L(w_1)^{\otimes k_1} \otimes \ldots \otimes L(w_{n-1})^{\otimes k_{n-1}}$ as $v_{w_1}^{\otimes k_1} \otimes \ldots \otimes v_{w_{n-1}}^{\otimes k_{n-1}}$ is the highest weight vector of weight λ (if v_{w_i} is the highest weight vector of $L(w_i)$). But $L(w_i) = \Lambda^i \mathbb{C}^n$ is a summand of $(\mathbb{C}^n)^{\otimes i}$, so $L(w_i)$ occurs in some $(\mathbb{C}^n)^{\otimes N}$, $N > 0$. Therefore, the crystal of $\mathbb{C}^n$ and the rule for the tensor product of a crystal determine the crystal for every representation $L(\lambda)$ of $\mathfrak{sl}_n$.

Now, we can introduce the *semi-standard Young tableau* of a representation (due to Hodge ($\sim$1930), Schur ($\sim$1900), and Young ($\sim$1900)). Write

$$B(w_1) = \boxed{1} \xrightarrow[1]{} \boxed{2} \xrightarrow[2]{} \boxed{3} \xrightarrow[3]{} \ldots \xrightarrow[n-1]{} \boxed{n}$$

for the crystal of the standard representation $\mathbb{C}^n$. Now, if $i < n$, denote

$$b_i = \boxed{1} \otimes \boxed{2} \otimes \ldots \otimes \boxed{i} \in B(w_1)^{\otimes i}.$$

The element b_i corresponds to the basis vector $v_1 \wedge v_2 \wedge \ldots \wedge v_i \in \Lambda^i \mathbb{C}^n$, where $\mathbb{C}^n$ has the basis $v_1, \ldots, v_n$.

Exercise 8.11:

(i) The vector b_i is a highest weight vector in $B(w_1)^{\otimes i}$ of weight $w_i = e_1 + \ldots + e_n$. (Recall that $b \in B$ is a highest weight vector if $\tilde{e}_i b = 0$ for all i). Hence, the connected component of $B(w_1)^{\otimes i}$ containing b_i is $B(w_i)$.

(ii) The connected component $B(w_i)$ consists precisely of

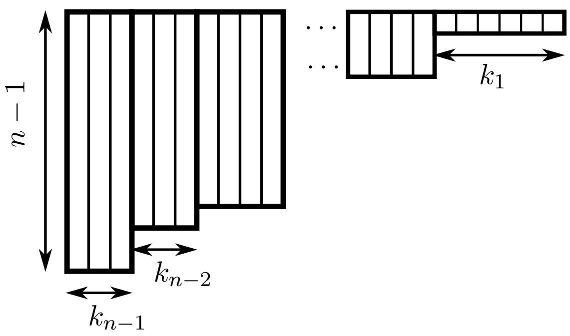
$$\{ \boxed{a_1} \otimes \boxed{a_2} \otimes \ldots \otimes \boxed{a_i} \mid 1 \leq a_1 < \ldots < a_i \leq n\} \subset B(w_1)^{\otimes i}.$$

We can write elements of the form $\boxed{a_1} \otimes \boxed{a_2} \otimes \ldots \otimes \boxed{a_i}$ as column vectors $\boxed{\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_i \end{array}}$, so the highest weight vectors are denoted $\boxed{\begin{array}{c} 1 \\ 2 \\ \vdots \\ i \end{array}}$.

Now, let $\lambda = \sum k_i w_i$ and embed $B(\lambda) \hookrightarrow B(w_1)^{\otimes k_1} \otimes \ldots \otimes B(w_{n-1})^{\otimes k_{n-1}}$ by mapping the highest weight vector $b_\lambda \mapsto b_1^{\otimes k_1} \otimes \ldots \otimes b_{n-1}^{\otimes k_{n-1}}$. Note that $b_1^{\otimes k_1} \otimes \ldots \otimes b_{n-1}^{\otimes k_{n-1}}$ actually is a highest weight vector in $B(w_1)^{\otimes k_1} \otimes \ldots \otimes B(w_{n-1})^{\otimes k_{n-1}}$. Now as $B(w_i) \hookrightarrow B(w_1)^{\otimes k_1}$ and hence

$$B(w_1)^{\otimes k_1} \otimes \ldots \otimes B(w_{n-1})^{\otimes k_{n-1}} \hookrightarrow B(w_1)^N, \quad N := \sum_{i=1}^{n-1} k_i,$$

we can represent any element in $B(w_1)^{\otimes k_1} \otimes \ldots \otimes B(w_{n-1})^{\otimes k_{n-1}}$ by a sequence of column vectors



where the entries are strictly increasing down columns, the length of the i-th row is $\sum_{j=i}^{n} k_j$. We say this young tableau has shape λ.

Definition 8.12: A *semi-standard Young tableaux* is an array of numbers as above, such that

(i) the numbers are strictly increasing down columns, and

(ii) decreasing along rows.

Theorem 8.13 (Exercise):

(i) The semi-standard Young tableau of shape λ are precisely elements of the connected component of $B(\lambda)$ in $B(w_1)^{\otimes k_1} \otimes \ldots \otimes B(w_{n-1})^{\otimes k_{n-1}}$.

(ii) Describe $\tilde{e}_i, \tilde{f}_i$ explicitly in terms of tableaux.

In the following, we will construct the Young tableau for the classical Lie algebras.

Example 8.14: $\mathfrak{so}_{2n+1}$: (Type B_n root systems) For the standard representation $\mathbb{C}^{2n+1}$ we have the crystal

$$\boxed{1} \xrightarrow[1]{} \boxed{2} \xrightarrow[2]{} \boxed{3} \xrightarrow[4]{} \cdots \xrightarrow[n-1]{} \boxed{n} \xrightarrow[n]{} \boxed{0} \xrightarrow[n]{} \boxed{\overline{n}} \xrightarrow[n-1]{} \cdots \xrightarrow[2]{} \boxed{\overline{2}} \xrightarrow[1]{} \boxed{\overline{1}}.$$

$\mathfrak{so}_{2n}$: (Type D_n root systems) For the standard representation $\mathbb{C}^{2n}$ we have the crystal

$$\begin{tikzcd}
 & & & & \boxed{n} \arrow[dr, "n"] & & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \boxed{\overline{2}} \arrow[r, "1"] & \boxed{\overline{1}}. \\
 & & & & \boxed{\overline{n}} \arrow[ur, "n-1"'] & & & &
\end{tikzcd}$$

$\mathfrak{sp}_{2n}$: (Type C_n root systems) For the standard representation $\mathbb{C}^{2n}$ we have the crystal

$$\boxed{1} \xrightarrow[1]{} \boxed{2} \xrightarrow[2]{} \boxed{3} \xrightarrow[4]{} \cdots \xrightarrow[n-1]{} \boxed{n} \xrightarrow[n]{} \boxed{\overline{n}} \xrightarrow[n-1]{} \cdots \xrightarrow[2]{} \boxed{\overline{2}} \xrightarrow[1]{} \boxed{\overline{1}}.$$

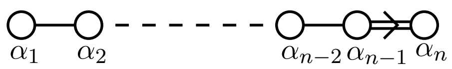
Exercise 8.15:

(i) Show that these are indeed the crystals of the standard representations of the classical Lie algebras.

(ii) What subcategory of the category of representations of $\mathfrak{g}$ do these representations generate? Consider the highest weight λ of the standard representation. This gives an element $\overline{\lambda} \in P/Q = Z(G)$, a finite group ($G$ is the simply connected group attached to $\mathfrak{g}$). Consider the subgroup $\langle\overline{\lambda}\rangle \leq P/Q$. We do not obtain all the representations unless P/Q is cyclic, generated by $\overline{\lambda}$. For the classical examples we have $P/Q = \mathbb{Z}/2 \times \mathbb{Z}/2$ for D_{2n}, $P/Q = \mathbb{Z}/4$ for D_{2n+1}, $P/Q = \mathbb{Z}/2$ for B_n and C_n.

(iii) (Optional) Write down a combinatorial set like Young tableaux that is the crystal of $B(\lambda)$ with λ obtained from the standard representation.

For B_n, we need one more representation, the *spin representation*. Recall that for B_n we had the dynkin diagram



Definition 8.16: The irreducible highest weight $\mathfrak{so}_{2n+1}$-representation $L(w_n)$, where w_n is the n-th fundamental weight, is called the *spin representation*.

Exercise 8.17: Use the Weyl dimension formula 7.58 to show that $\dim L(w_n) = 2^n$.

Define $B = \{(i_1, \dots, i_n) \mid i_j \in \{\pm 1\}\}$, $\mathrm{wt}(i_1, \dots, i_n) = \frac{1}{2}\sum_{j=1}^n i_j e_j \in P$, and for $1 \le j \le n-1$

$$\tilde{e}_j(i_1, \dots, i_n) = \begin{cases} (i_1, \dots, \underset{j}{+1}, \underset{j+1}{-1}, \dots, i_n) & \text{if } (i_j, i_{j+1}) = (-1, +1) \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{e}_n(i_1, \dots, i_n) = \begin{cases} (i_1, \dots, i_{n-1}, +1) & \text{if } i_n = -1 \\ 0 & \text{otherwise,} \end{cases}$$

so always $\tilde{e}_i^2 = 0$.

Fact 8.18: This is the crystal of the spin representation $L(w_n)$.

Remark 8.19: We have $\dim L(w_n) = \dim \Lambda^\bullet \mathbb{C}^n$. In fact, $\mathfrak{gl}_n \subset \mathfrak{so}_{2n+1}$, $A \mapsto \begin{pmatrix} A & & \\ & 0 & \\ & & -JA^TJ^{-1} \end{pmatrix}$, and $L(w_n)|_{\mathfrak{gl}_n} = \Lambda^\bullet \mathbb{C}^n$.

Exercise 8.20: Check that $B|_{\mathfrak{gl}_n}$ is a crystal of $\Lambda^\bullet \mathbb{C}^n$.

For type D_n, the situation is more complex. We can define representations

$$V^+ = L(w_n), \qquad V^- = L(w_{n-1}).$$

These are called *half-spin representations.* $B^\pm = \{(i_1, \dots, i_n) \mid i_j \in \{\pm 1\}, \pi_{i_j} = \pm 1\}$ + if B^+ and $-$ if B^-. wt, e_i, and f_i $(i < n)$ are defined as above, and

$$e_n(i_1, \dots, i_n) = \begin{cases} (i_1, \dots, i_{n-2}, +1, +1) & \text{if } (i_{n-1}, i_n) = (-1, -1) \\ 0 & \text{otherwise.} \end{cases}$$

8.2 Littelmann paths

Set $P_\mathbb{R} = P \otimes_\mathbb{Z} \mathbb{R}$. By a *path* we mean a piecewise linear continuous map $[0,1] \to P_\mathbb{R}$. We consider paths up to reparametrisation, i.e. $\pi \cong \pi \circ \phi$, where $\phi\colon [0,1] \to [0,1]$ is a piecewise-linear isomorphism.

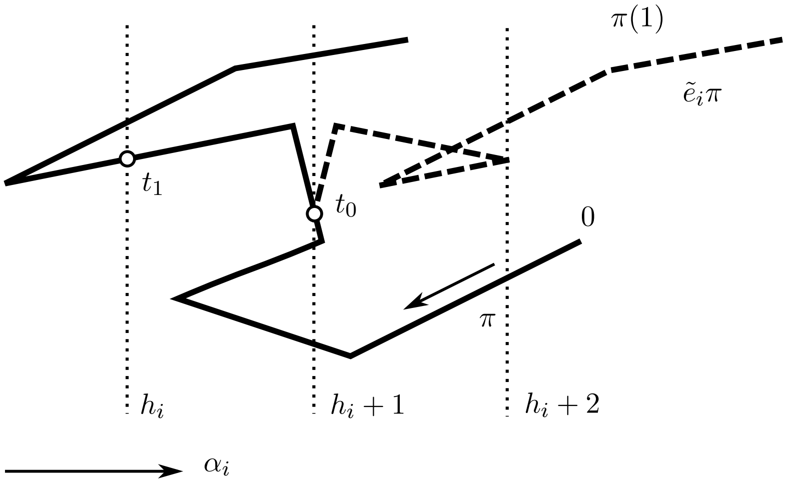
Let $\mathcal{P} = \{\text{paths } \pi \text{ s.t. } \pi(0) = 0,\, \pi(1) \in P\}$. We can define a crystal structure on $\mathcal{P}$. For $\pi \in \mathcal{P}$ define

$$\mathrm{wt}(\pi) = \pi(1).$$

To define $\tilde{e}_i(\pi)$, let

$$h_i = \min \mathbb{Z} \cap \{\langle \pi(t), \alpha_i^\vee \rangle \mid 0 \le t \le 1\} \le 0.$$

That is, h_i is the smallest integer in $\langle \alpha_i^\vee, \pi[0,1] \rangle$ (note that since $\pi(0) = 0$, we have $h_i \le 0$). If $h_i = 0$, set $\tilde{e}_i(\pi) = 0$ (this is not the path that stays at 0, but rather the extra element in the crystal). Otherwise $h_i < 0$, then take the smallest $t_1 > 0$ such that $\langle \pi(t_1), \alpha_i^\vee \rangle = h_i$ (i.e. the first time the path crosses h_i). Moreover, let t_0 be the largest $t_0 < t_1$ such that $\langle \pi(t_0), \alpha_i^\vee \rangle = h_i + 1$. We will define $\tilde{e}_i \pi$ as the path reflecting $\pi[t_0, t_1]$ in the hyperplane $\{\lambda \in P_\mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = h_i + 1\}$, and then translating $\pi[t_1, 1]$ while leaving $\pi[0, t_0]$ unchanged.



Expressed in formulas we have

$$\tilde{e}_i(\pi)(t) = \begin{cases} \pi(t), & \text{if } 0 \le t \le t_0 \\ \pi(t_0) + s_{\alpha_i}(\pi(t) - \pi(t_0)) = \pi(t) - \langle \pi(t) - \pi(t_0), \alpha_i^\vee \rangle \alpha_i, & \text{if } t_0 \le t \le t_1 \\ \pi(t) + \alpha_i, & \text{if } t \ge t_1. \end{cases}$$

Exercise 8.21: Show that $\varepsilon_i(\pi) = -h_i$.

Example 8.22: Let us compute some examples for $\mathfrak{sl}_2$:

$$\tilde{e}_i\left(\underset{-\frac{\alpha_i}{2}}{\bullet} \longleftarrow \underset{0}{\bullet} \right) = \underset{0}{\bullet} \longrightarrow \underset{\frac{\alpha_i}{2}}{\bullet},$$

$$\tilde{e}_i\left(\underset{0}{\bullet} \longrightarrow \underset{\frac{\alpha_i}{2}}{\bullet} \right) = 0, \qquad \text{and}$$

$$\tilde{e}_i\left(\underset{-\alpha_i}{\bullet} \longleftarrow \bullet \longleftarrow \underset{0}{\bullet} \right) = \underset{-\frac{\alpha_i}{2}}{\bullet} \rightleftarrows \underset{0}{\bullet},$$

$$\tilde{e}_i\left(\underset{-\frac{\alpha_i}{2}}{\bullet} \rightleftarrows \underset{0}{\bullet} \right) = \underset{0}{\bullet} \longrightarrow \bullet \longrightarrow \underset{\alpha_i}{\bullet},$$

$$\tilde{e}_i\left(\underset{0}{\bullet} \longrightarrow \bullet \longrightarrow \underset{\alpha_i}{\bullet} \right) = 0.$$

If π is a path, let $\pi^\vee$ be the reversed path, i.e. $t \mapsto \pi(1-t) - \pi(1)$. Define

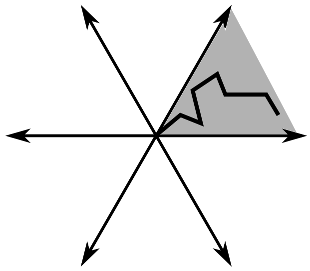
$$\tilde{f}_i(\pi) = (\tilde{e}_i(\pi^\vee))^\vee.$$

Exercise 8.23: $\mathcal{P}$ is a crystal with wt, $\tilde{e}_i$, $\tilde{f}_i$ defined as above.

Now, define

$$\mathcal{P}^+ = \left\{ \text{paths } \pi \text{ s.t. } \pi[0,1] \subset P_{\mathbb{R}}^+ = \{ x \in P_{\mathbb{R}} \mid \langle x, \alpha_i^\vee \rangle \ge 0 \ \forall i \} \right\}.$$

Observe that if $\pi \in \mathcal{P}^+$, then $\tilde{e}_i(\pi) = 0$ for all i.
For $\pi \in \mathcal{P}^+$ let B_π be the subcrystal of $\mathcal{P}$ generated by π, i.e. $B_\pi = \{\tilde{f}_{i_1} \tilde{f}_{i_2} \dots \tilde{f}_{i_r} \pi\}$.



Theorem 8.24 (Littelmann):

(i) If $\pi, \pi' \in \mathcal{P}^+$, then
$$B_\pi \cong B_{\pi'} \iff \pi(1) = \pi'(1)$$
(i.e. crystals with the same endpoint of highest weight paths are isomorphic).

(ii) There is a unique isomorphism of crystals $B(\pi(1)) \to B_\pi$ (where $B(\pi(1))$ is the crystal of the irreducible representation $L(\pi(1)))$ sending the highest weight $\pi(1)$ to a path π with endpoint $\pi(1)$.

Moreover, for paths of the form $\pi(t) = \lambda t$, $\lambda \in P^+$, Littlemann give an explicit combinatorically description of the paths in B_π.

Example 8.25 (Exercise):

(i) Consider $\mathfrak{sl}_3$ with simple roots α, β. We want to compute the crystal of the adjoint representation. First, show that
$$\tilde{e}_\beta \left(\underset{0}{\bullet} \longrightarrow \underset{-(\alpha+\beta)}{\bullet} \right) = \left(\underset{-\alpha}{\bullet} \longleftarrow \underset{0}{\bullet} \right),$$
and then compute the rest of the crystal and show that you obtain the adjoint representation of $\mathfrak{sl}_3$.

(ii) Consider the root system type G_2. You might have seen before that the smallest non-trivial representation is 7-dimensional. Compute the crystal for the 7-dimensional representation of G_2. Further, note that the second smallest non-trivial representation is 14-dimensional (the adjoint representation). Calculate the crystal for the 14-dimensional representation, and the tensor product of these two representations, if you feel like.

Remark 8.26: Littlemann's Theorem 8.24 allows us to define $B(\lambda)$ explicitly, without using $L(\lambda)$, and we can also prove Weyls character formula
$$\dim B(\lambda) = \frac{\sum_{w \in W} \det w e^{w}(\lambda + \rho) - \rho}{\prod (1 - e^{-\alpha})}$$
without the use of $L(\lambda)$. This gives a proof of the existence of crystals (Theorem 8.7) without quantum groups. To prove this, we can build $\operatorname{ch} L(\lambda)$, and indeed $L(\lambda)$ (and the crystal variants), one root at a time. This is called the *Demazure character formula.*

For every $w \in W$, there is an approximation to $L(\lambda)$ given by $L_w(\lambda)$ if $L_w(\lambda)$ is the $\mathfrak{n}^+$-submodule of $L(\lambda)$ generated by vectors $v_{w\lambda}$, where v_λ is the highest weight vector of $L(\lambda)$, and $v_{w\lambda}$ is the vector in $wL(\lambda)_\lambda$ (1-dimensional submodule).

Theorem 8.27 (Demazure character formula):

$$\operatorname{ch} L_w(\lambda) = D_w(e^\lambda),$$

where $w = s_{i_1} \cdot \ldots \cdot s_{i_r}$ is a reduced simple reflection decomposition of w (i.e. r minimal), and $D_w = D_{s_{i_1}} \cdot \ldots \cdot D_{s_{i_r}}$ with $D_{s_i} \colon \mathbb{Z}[P] \to \mathbb{Z}[P]$ defined by

$$\begin{aligned} D_{s_i}(f) &= \frac{f - s_i(f)}{1 - e^{-\alpha_i}} \\ &= (\operatorname{Id} + s_i)\left(\frac{f}{1 - e^{-\alpha_i}}\right) \\ &= \frac{1}{e^{\frac{\alpha_i}{2}} - e^{\frac{-\alpha_i}{2}}}(fe^{\frac{\alpha_i}{2}} - s_i(fe^{\frac{\alpha_i}{2}})), \quad \forall f \in \mathbb{Z}[P]. \end{aligned}$$

Note that

$$D_{s_i}(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda-\alpha_i} + \ldots + e^{s_i\lambda}, & \text{if } \langle \lambda, \alpha_i^\vee \rangle \geq 0, \\ 0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = -1, \\ -(e^{\lambda+\alpha_i} + \ldots + e^{s_i\lambda-\alpha_i}), & \text{if } \langle \lambda, \alpha_i^\vee \rangle < -1. \end{cases}$$

Additional sources

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